Intuitionistic monotone modal logic

Jim de Groot[1]

University of Bern, Bern, Switzerland
jim.degroot@unibe.ch

Introduction

Ever since Fitch first introduced a modal extension of intuitionistic logic [7], many different such logics have been put forward. These often differ in their connection between the modal operators, ranging from no connection at all [2, 16] to one modality being definable from the other [3], [2, Section 11]. The logic IK is one of myriad intuitionistic modal logics with a more subtle interaction between the modalities [6, 13, 5, 14]. It can be obtain by embedding modal logic into first order logic, and changing this first order logic to an intuitionistic one.

Akin to the normal modal case, there are various intuitionistic modal logics with monotone modalities [9, 15, 1, 4]. Interestingly, none of these is obtained from classical monotone modal logic [11] in the same way IK arises from normal modal logic. We close this gap by defining the intuitionistic modal logic $\mathsf{IM_s}$[1] as the set of formulas whose standard translation is derivable in a suitable intuitionistic first-order logic, and axiomatising the resulting logic.

Throughout this abstract, we denote by $\mathcal{L}$ the language generated by the grammar

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where p ranges over some set Prop of proposition letters. This can be viewed as the language underlying classical and intuitionistic modal logic with normal or monotone modalities. The first step in our quest for $\mathsf{IM_s}$ is to define a suitable first-order logic FOM that describes (classical) monotone modal logic.

Going monotone

Monotone modal logic M is the extension of classical propositional logic with a modal operator $\Box$ that satisfies $(p \to q)/(\Box p \to \Box q)$. We can define $\Diamond = \neg\Box\neg$, which will be monotone as well. It can be interpreted in so-called neighbourhood models.

Definition. A *neighbourhood model* is a tuple (W, γ, V) consisting of a nonempty set W, a *neighbourhood function* $\gamma : W \to \mathcal{PP}W$, an a valuation $V : \mathrm{Prop} \to \mathcal{P}W$. The modal operators can be interpreted via

$$\mathfrak{M}, w \Vdash \Box\varphi \quad \text{iff} \quad \text{there exists } a \in \gamma(w) \text{ such that for all } v \in a,\ \mathfrak{M}, v \Vdash \varphi$$
$$\mathfrak{M}, w \Vdash \Diamond\varphi \quad \text{iff} \quad \text{for all } a \in \gamma(w) \text{ there exists } v \in a \text{ such that } \mathfrak{M}, v \Vdash \varphi$$

Taking a first-order perspective of neighbourhood models is not as easy as for Kripke models, because γ relates worlds to sets of worlds. To get around this, we use a two-sorted language, with one sort representing the worlds and the other the neighbourhoods [8, 12, 10].

Definition. Let FOM be the two-sorted first-order logic with sorts world and nbhd, a predicate N between world and nbhd, a predicate E between nbhd and world, and a unary predicate P_i of type world for each $p_i \in \mathrm{Prop}$.

[1] The analogy of our logic with IK suggests naming it IM, but this is already used in [3, 4] for different logics. To avoid confusion, we adopt the name $\mathsf{IM_s}$. The "s" indicates that our logic is stronger than the logic IM.

Then a FOM-structure is a tuple $\mathfrak{M} = (D_{\mathsf{w}}, D_{\mathsf{n}}, N, E, P_i)$ consisting of sets D_{w} and D_{n} interpreting the sorts, relations $N \subseteq D_{\mathsf{w}} \times D_{\mathsf{n}}$ and $E \subseteq D_{\mathsf{n}} \times D_{\mathsf{w}}$, and subsets $P_i \subseteq D_{\mathsf{w}}$ for each $p_i \in \mathrm{Prop}$. Every neighbourhood model $\mathfrak{M} = (W, \gamma, V)$ gives rise to a FOM-structure

$$\mathfrak{M}^{\circ} = \big(W, \bigcup\{\gamma(w) \mid w \in W\}, R_{\gamma}, R_{\ni}, \{V(p_i) \mid p_i \in \mathrm{Prop}\}\big),$$

where $wR_{\gamma}a$ iff $a \in N(w)$ and $aR_{\ni}w$ iff $w \in a$. In the converse direction, a FOM-structure $\mathfrak{M} = (D_{\mathsf{w}}, D_{\mathsf{n}}, N, E, P_i)$ yields a neighbourhood model $\mathfrak{M}^{\bullet} = (D_{\mathsf{w}}, \gamma, V)$, where $V(p_i) = P_i$ and

$$\gamma(x) = \{\{y \in D_{\mathsf{w}} \mid aEy\} \mid xNa\}.$$

We can define the standard translation $\mathrm{st}_x : \mathcal{L} \to \mathsf{FOM}$ by

$$\begin{aligned} \mathrm{st}_x(\top) &= (x = x) & \mathrm{st}_x(\varphi \wedge \psi) &= \mathrm{st}_x(\varphi) \wedge \mathrm{st}_x(\psi) \\ \mathrm{st}_x(\bot) &= (x \neq x) & \mathrm{st}_x(\varphi \vee \psi) &= \mathrm{st}_x(\varphi) \vee \mathrm{st}_x(\psi) \\ \mathrm{st}_x(p_i) &= \mathrm{st}\, P_i x & \mathrm{st}_x(\varphi \to \psi) &= \mathrm{st}_x(\varphi) \to \mathrm{st}_x(\psi) \\ \mathrm{st}_x(\Box\varphi) &= \exists a.xNa \wedge \forall y.aEy \to \mathrm{st}_y(\varphi) & \mathrm{st}_x(\Diamond\varphi) &= \forall a.xNa \to \exists y.aEy \wedge \mathrm{st}_y(\varphi) \end{aligned}$$

Proposition. *For all $\varphi \in \mathcal{L}$, neighbourhood models $\mathfrak{M}$ and FOM-structures $\mathfrak{N}$, we have:*

$$\mathfrak{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}^{\circ} \models \mathrm{st}_x(\varphi)[w], \qquad \mathfrak{N}^{\bullet}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{N} \models \mathrm{st}_x(\varphi)[w].$$

Going intuitionistic

We now change the first-order logic FOM to the intuitionistic first-order logic IFOM of the same signature. Note that st_x can be viewed as a translation $\mathrm{st}_x : \mathcal{L} \to \mathsf{IFOM}$. Then we can define:

Definition. $\mathsf{IM_s} = \{\varphi \in \mathcal{L} \mid \mathsf{IFOM} \models \mathrm{st}_x(\varphi)\}$.

We can now ask how to axiomatise $\mathsf{IM_s}$. It turns out that we can do so as follows:

Definition. Let $\mathsf{IM_{Ax}}$ be the smallest set of $\mathcal{L}$-formulas that contains an axiomatisation for intuitionistic logic as well as the axioms

$$\Box(p \wedge q) \to \Box p, \qquad \Diamond(p \wedge q) \to \Diamond p, \qquad (\Box p \wedge \Diamond\neg p) \to \bot \quad \text{and} \quad (\Box\top \to \Diamond p) \to \Diamond p,$$

and that is closed under modus ponens, uniform substitution, and the congruence rules

$$\frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q} \quad \text{and} \quad \frac{p \leftrightarrow q}{\Diamond p \leftrightarrow \Diamond q}.$$

We write $\mathsf{IM_{Ax}} \vdash \varphi$, and say that φ is *derivable*, if φ is in $\mathsf{IM_{Ax}}$.

Alternatively, we can replace the monotonicity axioms and congruence rules for the *monotonicity rules* $(p \to q)/(\Box p \to \Box q)$ and $(p \to q)/(\Diamond p \to \Diamond q)$ to obtain the same logic.

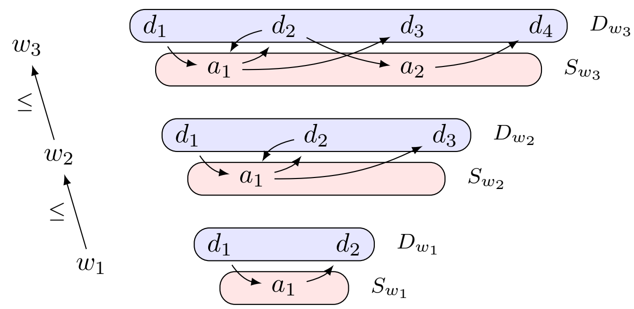
Our main theorem reads:

Theorem 1. *For any $\varphi \in \mathcal{L}$, we have $\varphi \in \mathsf{IM_s}$ if and only if $\mathsf{IM_{Ax}} \vdash \varphi$.*

Going semantic

One way to prove Theorem 1 is via a semantic detour. This not only allows us to use a routine canonical model construction, but also exposes a new way of thinking about neighbourhood models in an intuitionistic setting. Theorem 1 follows from the following three steps.

Step 1. Use IFOM***-structures as a semantics for*** $\mathsf{IM_s}$***.*** Since $\mathsf{IM_s}$ is defined via the first-order logic IFOM, we can use the first-order structures for IFOM as a semantics for $\mathsf{IM_s}$. An IFOM-structure consists of a poset $(W, \leq)$ with at each world $w \in W$ a classical first-order structure, such that these classical structures increase along $\leq$. For example:



We can then interpret $\mathcal{L}$-formulas at pairs $\langle w, d \rangle$ where $d \in D_w$. We let $\langle w, d \rangle \Vdash \varphi$ if $w \models \mathrm{st}_x(\varphi)[d]$. Concretely, we let $\langle w, d \rangle \Vdash p_i$ if d is in the interpretation of P_i at w, intuitionistic connectives are interpreted as usual in first-order intuitionistic logic, and:

$$\begin{array}{lll} \langle w, d \rangle \Vdash \Box\varphi & \text{iff} & \text{there exists } a \in A_w \text{ such that } dNa \text{ and} \\ & & \text{for all } w' \geq w \text{ and } d' \in D_{w'},\ aE_{w'}d' \text{ implies } \langle w, d' \rangle \Vdash \varphi \\ (w, x) \Vdash \Diamond\varphi & \text{iff} & \text{for all } w' \geq w \text{ and all } a' \in D_n(w'), \\ & & \text{if } xN_{w'}a' \text{ then there exists } y' \in D_s(w') \text{ such that } a'E_{w'}y' \text{ and } (w', y') \Vdash \varphi \end{array}$$

Let us write $\mathsf{FOS} \Vdash \varphi$ if φ is true in all worlds of all such first-order structures. Then by definition of $\mathsf{IM_s}$ and IFOM we have:

Proposition. *For all* $\varphi \in \mathcal{L}$*, we have* $\varphi \in \mathsf{IM_s}$ *if and only if* $\mathsf{FOS} \Vdash \varphi$.

Step 2. Define intuitionistic neighbourhood models. Taking inspiration from the first-order semantics, we see that an intuitionistic version of a neighbourhood can change when moving along the intuitionistic accessibility relation. Guided by this, we define an *intuitionistic neighbourhood* as a partial function

$$a : W \rightharpoonup \mathcal{P}W$$

such that $\mathrm{dom}(a) = \{w \in W \mid a(w) \text{ is defined}\}$ is upward closed in $(W, \leq)$. An *intuitionistic neighbourhood model* is then given by a tuple $(W, \leq, N, V)$ such that $(W, \leq)$ is a poset, V is a valuation that assigns to each proposition letter an upset of $(W, \leq)$, and N is a collection of intuitionistic neighbourhoods. The modalities are then interpreted as

$$\begin{array}{lll} \mathfrak{M}, w \Vdash \Box\varphi & \text{iff} & \text{there exists } a \in N \text{ such that } w \in \mathrm{dom}(a) \text{ and} \\ & & \text{for all } w' \geq w,\ v \in a(w') \text{ implies } v \Vdash \varphi \\ \mathfrak{M}, w \Vdash \Diamond\varphi & \text{iff} & \text{for all } w' \geq w \text{ and all } a \in N \text{ such that } w' \in \mathrm{dom}(a) \\ & & \text{there exists } v \in a(w') \text{ such that } v \Vdash \varphi \end{array}$$

We write INM for the class of intuitionistic neighbourhood models, and $\mathsf{INM} \Vdash \varphi$ if φ is valid in all intuitionistic neighbourhood models. A routine canonical model construction now proves:

Proposition. *For all* $\varphi \in \mathcal{L}$*, we have* $\mathsf{IM_{Ax}} \vdash \varphi$ *iff* $\mathsf{INM} \Vdash \varphi$.

Step 3. Translate between sematnics. Any IFOM-structure gives rise to an intuitionistic neighbourhood model whose worlds are given by pairs $\langle w, d \rangle$ where w is a world and $d \in D_w$. Conversely, we can transform intuitionistic neighbourhood models into IFOM-structures. This requires an additional unravelling-like construction which allows us to construct the required structure of a poset with a domain for each element. If we do so carefully, we can prove:

Proposition. *For all $\varphi \in \mathcal{L}$, we have* $\mathsf{INM} \Vdash \varphi$ *if and only if* $\mathsf{FOS} \Vdash \varphi$.

Conclusion and further work

We have given an intuitionistic monotone modal logic obtained from classical monotone modal logic via a first-order route. This opens up many avenues for further research. For example, it would be interesting to study the connection with existing intuitionistic (monotone) modal logics. Also, the first-order perspective given in this abstract may also be used to find an intuitionistic non-monotone modal logic.

References

[1] G. Bellin, V. de Paiva, and E. Ritter. Extended Curry-Howard correspondence for a basic constructive modal logic. In *Methods for Modalities II*, 2001.

[2] M. Božić and K. Došen. Models for normal intuitionistic modal logics. *Studia Logica*, 43:217–245, 1984.

[3] R. A. Bull. Some modal calculi based on IC. In J.N. Crossley and M.A.E. Dummett, editors, *Formal Systems and Recursive Functions*, pages 3–7, Amsterdam, North Holland, 1965. Elsevier.

[4] T. Dalmonte, C. Grellois, and N. Olivetti. Intuitionistic non-normal modal logics: A general framework. *Journal of Philosophical Logic*, 49:833–882, 2020.

[5] W. B. Ewald. Intuitionistic tense and modal logic. *Journal of Symbolic Logic*, 51(1):166–179, 1986.

[6] G. Fischer Servi. Semantics for a class of intuitionistic modal calculi. In D. Chiara and M. Luisa, editors, *Italian Studies in the Philosophy of Science*, pages 59–72, Netherlands, 1980. Springer.

[7] F. B. Fitch. Intuitionistic modal logic with quantifiers. *Portugaliae mathematica*, 7(2):113–118, 1948.

[8] J. Flum and M. Ziegler. *Topological Model Theory.* Springer, Berlin, Heidelberg, 1980.

[9] R. I. Goldblatt. Grothendieck topology as geometric modality. *Mathematical Logic Quarterly*, 27:495–529, 1981.

[10] J. de Groot. Hennessy-Milner and Van Benthem for instantial neighbourhood logic. *Studia Logica*, 110:717–743, 2022.

[11] H. H. Hansen. Monotonic modal logics. Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2003. illc-MoL:2003-24.

[12] H. H. Hansen, C. Kupke, and E. Pacuit. Neighbourhood structures: bisimilarity and basic model theory. *Logical Methods in Computer Science*, 5(2), 2009.

[13] G. Plotkin and C. Stirling. A framework for intuitionistic modal logics: Extended abstract. In *Proc. TARK 1986*, pages 399–406, San Francisco, USA, 1986. Morgan Kaufmann Publishers Inc.

[14] A. K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic.* PhD thesis, University of Edinburgh, 1994.

[15] D. Wijesekera. Constructive modal logics I. *Annals of Pure and Applied Logic*, 50:271–301, 1990.

[16] F. Wolter and M. Zakharyaschev. Intuitionistic modal logic. In A. Cantini, E. Casari, and P. Minari, editors, *Logic and Foundations of Mathematics: Selected Contributed Papers of the Tenth International Congress of Logic, Methodology and Philosophy of Science*, pages 227–238, Netherlands, 1999. Springer.