On some properties of Płonka sums

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1 Introduction

The Płonka sum is a construction introduced in the 1960s in Universal Algebra by the eponymous Polish mathematician [7] (see also [9, 1]) that allows to construct a new algebra out of a semilattice direct system of similar (disjoint) algebras, called the fibers (of the system). The theory of Płonka sums has been mostly studied in the case of a similarity type without constant functional symbols: in such a case the fibres are subalgebras of their Płonka sum.
Płonka sums are strictly connected with regular identities. Recall that an identity $\alpha \approx \beta$ (in an algebraic language τ and over some set of variables X) is *regular* if $Var(\alpha) = Var(\beta)$. An identity $\alpha \approx \beta$ is valid in the Płonka sum over a non-trivial semilattice direct system $\mathbb{A} = ((\mathbf{A}_i)_{i \in I}, (I, \leq), (p_{ij})_{i \leq j})$ (i.e. $|I| \geq 2$) if and only if it is a regular identity valid in each of the fibers of $\mathbb{A}$.
Given a class of similar algebras $\mathcal{K}$, its *regularization* is the variety $\mathcal{R}(\mathcal{K})$ defined by the regular identities valid in $\mathcal{K}$. This variety is particularly interesting when the class $\mathcal{K}$ is a strongly irregular τ-variety $\mathcal{V}$ - an assumption that includes almost all examples of known irregular varieties -, i.e. a variety satisfying an identity of the form $p(x, y) \approx x$ for some binary τ-term p: in such a case, every algebra in $\mathcal{R}(\mathcal{V})$ is the Płonka sum over a semilattice direct systems (with zero) of algebras in $\mathcal{V}$.
The following is quite natural.
Question: which algebraic properties holding for $\mathcal{V}$ are also valid for $\mathcal{R}(\mathcal{V})$?

With respect to the question, several properties for $\mathcal{R}(\mathcal{V})$ have been established over the years, including the description of the lattice of the subvarieties of regularized varieties [3], of their subdirectly irreducible members [5], the equational basis of regularized varieties [11], and the structure of free algebras [10].
In this talk, we will give a brief overview of the theory of Płonka sums over an algebraic language *with constant symbols* with a particular emphasis on the structural side. Then we will address the above question with respect to the following properties: local finiteness, epimorphism surjectivity (ES), amalgamation property (AP) and congruence extension property (CEP).

2 Free algebras and local finiteness

Free algebras in the regularization $R(\mathcal{V})$ (of a strongly irregular variety $\mathcal{V}$) are characterized by Romanowska in [10] under the assumption that the language of $\mathcal{V}$ contains no constant symbols (see also [8]). The following covers the more general case of an algebraic language containing constants' symbols.

Theorem 1. *Let $\mathbf{A}$ be an algebra in $\mathcal{R}(\mathcal{V})$, with $\mathcal{V}$ a strongly irregular variety. Then $\mathbf{A} \in \mathcal{R}(\mathcal{V})$ is free on the set of generators $\{a_j\}_{j \in J}$ iff $\mathbf{A}$ is a Płonka sum over a semilattice direct sistem with zero $((\mathbf{A}_i)_{i \in I}, (I, \leq), (p_{ij})_{i \leq j})$ such that*

1. *$(I, \leq)$ is the free semilattice with zero on the set of generators J;*

2. *for every $i \in I, \mathbf{A}_i$ is a finitely generated $\mathcal{V}$-free algebra.*
   In particular $\mathbf{A}_i$ has n generators $G^i = \{g_1^i, \ldots, g_n^i\}$ iff $i = i_1 \vee \cdots \vee i_n$, for $i_1, \ldots, i_n \in J$ ($i_k \neq i_m$ for $k \neq m$), and if $i \in J$ then $\mathbf{A}_i$ is one generated by the element $g_1^i = a_{i_1} = a_i$;

3. *for every $i \in I$, $p_{i_0 i}$ is the unique homomorphism from $\mathbf{A}_{i_0}$ into $\mathbf{A}_i$, where i_0 is the least element of I, while for every $i_1, ..., i_n \in J$ ($\forall n \in \mathbb{N}^+$) if $i = i_1 \vee ... \vee ... \vee i_n$ then for every $j \in \{1, ..., n\} : p_{i_j i} : \mathbf{A}_{i_j} \to \mathbf{A}_i$ is the unique (injective) homomorphism extending the map $p^0_{i_j i} \colon \{a_{i_j}\} \to \mathbf{A}_i$, $a_{i_j} \mapsto p^0_{i_j i}(g^i_j) := g^k_j$. In particular, for $i_0 \neq i \leq k$, $p_{ik} \colon \mathbf{A}_i \to \mathbf{A}_k$ is the unique (injective) homomorphism extending the map $p^0_{ik} \colon G^i \to \mathbf{A}_k$, $g^i_j \mapsto p^0_{ik}(g^i_j) := g^k_j$.*

Corollary 1. *Let $\mathcal{V}$ be a strongly irregular variety, $\mathcal{R}(\mathcal{V})$ its regularization, $\mathbf{A}_n$ the $\mathcal{R}(\mathcal{V})$-free algebra on $n \in \mathbb{N}$ generators, then*

$$|A_n| = \sum_{j=0}^{n} \binom{n}{j} |B_j|,$$

where $\mathbf{B}_j$ is the $\mathcal{V}$-free algebra on j generators.

Since local finiteness can be controlled on free algebras [2, Theorem 10.15], the following corollary holds.

Corollary 2. *Let $\mathcal{V}$ be a strongly irregular variety. If $\mathcal{V}$ is locally finite, then $\mathcal{R}(\mathcal{V})$ is locally finite.*

3 Congruences

In [1] a very natural **problem** is posed: to describe the congruence lattice of algebras in regular varieties.

Let $\mathbb{A} = ((\mathbf{A}_i)_{i \in I}, (I, \leq), (p_{ij})_{i \leq j})$ be a semilattice direct system in a strongly irregular τ-variety $\mathcal{V}$ and $\mathbf{A}$ its Płonka sum. Let's begin our investigation by starting with a congruence and trying to deduce its essential structural features.

Let $\theta \in Con(\mathbf{A})$, then for every $(i, j) \in I \times$ I we define $\theta_{ij} := \theta \cap (A_i \times A_j)$ and $S_\theta := \{(i, j) \in I \times I \mid \theta_{ij} \neq \emptyset\}$.

Lemma 1. *Let τ be any algebraic language, then $\forall (i, j) \in S_\theta, \forall a \in A_i : (a, p_{ii \vee j}(a)) \in \theta$. Moreover, S_θ is a reflexive and symmetric subsemilattice of $\mathbf{I} \times \mathbf{I}$.*

Unfortunately, *transitivity is not guaranteed*, but a (kind of) *weak* form of *transitivity*, outlined in the following Lemma, is always valid.

Lemma 2. *Let τ be any algebraic language, then $\forall i, j, k \in I : (i, j), (j, k) \in S_\theta \Rightarrow (i, i \vee k) \in S_\theta$.*

To simplify the exposition, we will say that S_θ is **upper transitive**.
In some particular, yet relevant, cases, S_θ turns out to be a congruence on $\mathbf{I}$.

Corollary 3. *Let τ be any algebraic language. If one of the following occurs:*

(i) $\mathbf{I}$ *is a chain;*

(ii) τ *be an algebraic language containing constants*

then $S_\theta \in Con(\mathbf{I})$.

Consequently, transitivity is always ensured for algebraic languages having constants.

The following result provides the sought-after characterization.

Theorem 2. *Let* τ *be any algebraic language. Let* $S \subseteq I \times I$ *and* $(\theta_{ii})_{i\in I}$ *be a family such that the following conditions occur:*

(i) S *is a reflexive, symmetric and upper transitive subsemilattice of* $\mathbf{I} \times \mathbf{I}$*;*

(ii) $\forall i \in I : \theta_{ii} \in Con(\mathbf{A}_i)$*;*

(iii) $\forall (i,j) \in I \times I : \theta_{ii} \subseteq (p_{ii\vee j} \times p_{ii\vee j})^{-1}(\theta_{i\vee j, i\vee j})$*, with equality if* $(i,j) \in S$*;*

(iv) $\forall (i,j) \in I \times I : (i,j) \in S \iff (i, i\vee j), (j, i\vee j) \in S, (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j}) \neq \emptyset$

For every $(i,j) \in S \setminus \Delta_{\mathbf{I}}$*, let* $\theta_{ij} := (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j})$*, then*

$$\theta := \bigcup_{(i,j)\in S} \theta_{ij} \in Con(\mathbf{A}).$$

Furthermore, all the elements of $Con(\mathbf{A})$ *arise in this way.*

In the case of an algebraic language containing constants (or if $\mathcal{V}$ admits an algebraic constant), the characterization takes on a simpler form, since requirement *(iv)* is automatically satisfied and $S \in Con(\mathbf{I})$.

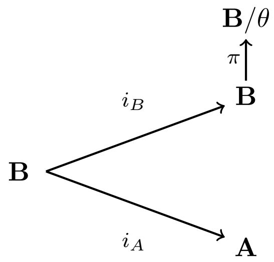
4 Amalgamation and Congruence Extension Property

It is very natural to ask whether the amalgamation property (AP) can be "lifted" through Płonka sums. More precisely, does $\mathcal{R}(\mathcal{V})$ have the (strong) AP when $\mathcal{V}$ is strongly irregular and has (strong) AP?
The fact that semilattices (with zero) have (strong) AP could point to a positive answer. Surprisingly enough, Hall [4, Remark 5] showed that Clifford semigroups, namely the regularization of groups (see [9] for details), fail to have AP.
Thanks to the following notion, Hall's argument can be easily generalized.

Definition 1. An algebra $\mathbf{A}$ is *hereditarily simple* if each of its subalgebras is simple.
A variety $\mathcal{V}$ is *hereditarily simple* if each simple algebra in $\mathcal{V}$ is hereditarily simple.

The following result by Pastijn [6] links the validity of the congruence extension property in a strongly irregular variety to the existence of amalgams in $\mathcal{R}(\mathcal{V})$ for specific V-formations in $\mathcal{R}(\mathcal{V})$.

Proposition 1 ([6])**.** *Let* $\mathcal{V}$ *be a strongly irregular variety. Then* $\mathcal{V}$ *has CEP iff* $\forall \mathbf{A} \in \mathcal{V}, \forall \mathbf{B} \leq \mathbf{A}, \forall \theta \in Con(\mathbf{B})$ *the following* V*-formation in* $\mathcal{R}(\mathcal{V})$



has an amalgam in $\mathcal{R}(\mathcal{V})$*.*

Corollary 4. *Let* $\mathcal{V}$ *be a strongly irregular variety. If* $\mathcal{R}(\mathcal{V})$ *has* AP*, then* $\mathcal{V}$ *has* CEP*. In particular, if* $\mathcal{V}$ *is not hereditarily simple, then* $\mathcal{R}(\mathcal{V})$ *fails to have* AP*.*

Pastijn [6] indeed gives an answer to our question with respect to (strong) AP and CEP.

Theorem 3 ([6])**.** *Let* $\mathcal{V}$ *be a strongly irregular variety. Then:*

1. $\mathcal{R}(\mathcal{V})$ *has* CEP *if and only if* $\mathcal{V}$ *has* CEP*.*
2. $\mathcal{R}(\mathcal{V})$ *has (strong)* AP *if and only if* $\mathcal{V}$ *has* CEP *and (strong)* AP*.*

5 Epimorphism Surjectivity

Epimorphism surjectivity is another property preserved by Płonka sums. More specifically

Theorem 4. *Let* $\mathcal{V}$ *be a strongly irregular variety.* $\mathcal{R}(\mathcal{V})$ *has* ES *if and only if* $\mathcal{V}$ *has* ES*.*

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