Canonical model construction for a real-valued modal logic

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Introduction. Abelian logic, introduced in [2, 6], is the logic of lattice-ordered abelian groups, or, equivalently, $\mathbb{R}$ equipped with the operations min, max, $+$, $-$, and 0. A minimal modal extension of this logic, called Abelian modal logic, was defined in [3] based on standard Kripke frames, where the operations on $\mathbb{R}$ are calculated locally at worlds and the modal operator $\Box$ is interpreted by taking infima of values at accessible worlds. Abelian modal logic not only provides a framework for reasoning about (transitions between) states represented by vectors over $\mathbb{R}$, but also contains, under translation, the minimal Łukasiewicz modal logic studied in [4].

Notably, both Abelian modal logic and Łukasiewicz modal logic lack an explicit finitary axiomatisation. As a first step towards addressing this gap, an axiomatization was obtained in [3] for the modal-multiplicative fragment of the logic following a proof-theoretic approach. In [5], this approach was used to obtain a quasi-equational axiomatization of the equational theory of the modal-meet-semilattice-ordered-monoid fragment. In this work, we employ a canonical model construction to establish completeness of a quasi-equational axiomatization for a variation on the latter logic. In particular, instead of considering truth values in $\mathbb{R}$, we restrict our attention to the set $\mathbb{R}_{\leq 0}$ of non-negative real numbers, where 0 is considered the designated truth value, and the strictly negative numbers represent increasing degrees of falsehood. Alternatively, truth values can be taken from the open-closed unit interval $(0, 1]$, with the operations min, $\cdot$ (multiplication), and 1.

Semantics. Formulas $\varphi, \psi, \chi, \ldots$ are defined over a countably infinite set of propositional variables P with respect to a language with binary operation symbols $\wedge$ and $\oplus$, unary operation symbol $\Box$, and constant symbol e. We also define $0\varphi \coloneqq \mathrm{e}$ and $(n+1)\varphi \coloneqq n\varphi \oplus \varphi$ for $n \in \mathbb{N}$. An equation is an ordered pair of formulas, written $\varphi \approx \psi$, and $\varphi \leq \psi$ abbreviates $\varphi \wedge \psi \approx \varphi$.

Let $\mathbf{R}_{\leq 0}$ be the algebra $\langle \mathbb{R}_{\leq 0}, \min, +, 0 \rangle$. A $\mathsf{K}(\mathbf{R}_{\leq 0})$-model $\mathfrak{M} = \langle W, R, V \rangle$ consists of a non-empty set of worlds W, an accessibility relation $R \subseteq W^2$, and an evaluation map V that assigns to each $p \in P$ a function $V(p) \colon W \to \mathbb{R}_{\leq 0}$. The value $[\![\varphi]\!](w)$ of a formula φ in $\mathfrak{M}$ at a world $w \in W$ is defined recursively as follows:

$$\begin{aligned}
[\![p]\!](w) &= V(p)(w) \\
[\![\mathrm{e}]\!](w) &= 0 \\
[\![\psi \oplus \chi]\!](w) &= [\![\psi]\!](w) + [\![\chi]\!](w) \\
[\![\psi \wedge \chi]\!](w) &= \min([\![\psi]\!](w), [\![\chi]\!](w)) \\
[\![\Box\psi]\!](w) &= \bigwedge\{[\![\psi]\!](v) \mid wRv\}.
\end{aligned}$$

Note that the meet in the interpretation of boxed formulas does not exist when the set contains arbitrarily small negative real numbers, and we therefore restrict our attention to models where this issue does not occur, i.e., models such that $[\![\varphi]\!](w)$ is defined for every formula φ and world $w \in W$. Note also that the empty meet is well-defined and equal to 0.

For an equation $\varphi \approx \psi$, we define

$$\models_{\mathsf{K}(\mathbf{R}_{\leq 0})} \varphi \approx \psi \;:\Longleftrightarrow\; [\![\varphi]\!](w) = [\![\psi]\!](w) \text{ for every } \mathsf{K}(\mathbf{R}_{\leq 0})\text{-model } \langle W, R, V \rangle \text{ and } w \in W.$$

It is easily proved, along the same lines as the proof for Abelian modal logic in [3], that this logic admits a finite model property; that is, $\models_{\mathsf{K}(\mathbf{R}_{\leq 0})} \varphi \approx \psi$ if and only if $[\![\varphi]\!](w) = [\![\psi]\!](w)$ for every *finite* $\mathsf{K}(\mathbf{R}_{\leq 0})$-model $\langle W, R, V\rangle$ and $w \in W$. Decidability is also easily established, e.g., by providing a tableau proof system, again following the methodology of [3].

Axiomatisation. Next we define the target axiomatisation for our modal logic. Let $\mathcal{Q}_{\mathrm{msm}}$ denote the quasi-variety of algebras $\langle A, \wedge, \oplus, \mathrm{e}, \Box\rangle$ defined by equations axiomatizing the variety of meet-semilattice-ordered commutative monoids, together with the (quasi-)equations

- $x \leq \mathrm{e}$ (e is the greatest element);
- $x \oplus z \leq y \oplus z \implies x \leq y$ (cancellation);
- $nx \leq ny \implies x \leq y$ for each $n \in \mathbb{N}^+$ (torsion-freeness);
- $\Box x \oplus \Box y \leq \Box(x \oplus y)$;
- $\Box(nx) \approx n(\Box x)$ for each $n \in \mathbb{N}$.

We prove the following:

Theorem 1 (Completeness theorem)**.** *For any equation $\varphi \approx \psi$,*

$$\models_{\mathsf{K}(\mathbf{R}_{\leq 0})} \varphi \approx \psi \quad\iff\quad \mathcal{Q}_{\mathrm{msm}} \models \varphi \approx \psi.$$

Canonical model construction. Recall that in classical modal logic the canonical model has as its worlds maximally consistent sets of formulas, or, equivalently, ultrafilters of the free Boolean algebra [1]. These are in one-to-one correspondence with homomorphisms from the free Boolean algebra to the two-element Boolean algebra. Let $\mathbf{F}_\Box$ denote the free algebra of $\mathcal{Q}_{\mathrm{msm}}$ over the set of generators P, and $\mathbf{F}$ its non-modal reduct. Analogously to the classical case, we define the set of worlds of our canonical model to be the set of homomorphisms from $\mathbf{F}$ to $\mathbf{R}_{\leq 0}$. We define an accessibility relation R on this set as follows:

$$h_1 R h_2 \;:\Longleftrightarrow\; h_1(\Box a) \leq h_2(a) \text{ for each } a \in F,$$

i.e., for h_2 to be a successor of h_1 we request that h_2 internally "thinks" each a to be more true than what h_1 "thinks" about the boxed version $\Box a$. The valuation is standard: the value of $p \in P$ in a world h is defined to be $h(p)$; that is, $V(p)(h) := h(p)$.

We need two main results in order to establish the completeness theorem: a truth lemma and what we will call a separation lemma. As usual, the truth lemma states that the values a world h internally assigns to elements of $\mathbf{F}$ (essentially formulas), equals the "external" truth-value of the associated formulas in that world h. The separation lemma resembles the Lindenbaum lemma: given two distinct elements of $\mathbf{F}$ (essentially two non-equivalent formulas), we need to produce a world h that separates these, i.e., assigns them distinct values.

In both cases, we construct a homomorphism $h\colon \mathbf{F} \to \mathbf{R}_{\leq 0}$ by first defining a homomorphism g from a certain freely generated non-modal algebra $\mathbf{A}$ to $\mathbf{R}_{\leq 0}$. This $\mathbf{A}$ is defined in such a way that it admits $\mathbf{F}$ as a quotient. By ensuring that g factors through this quotient map, we find the desired homomorphism h. The proof that there exists a morphism g with the required properties makes use of Farkas' lemma from linear algebra. This is combined with topological techniques to reduce infinite sets of requirements to finite ones that differ between the two lemmata.

Lemma 2 (Separation lemma)**.** *Let $a, b \in \mathbf{F}_\Box$ with $a \not\leq b$. Then there exists a homomorphism $h\colon \mathbf{F} \to \mathbf{R}_{\leq 0}$ such that $h(a) > h(b)$.*

In particular, if $a \neq b$, then there exists a homomorphism $h\colon \mathbf{F} \to \mathbf{R}_{\leq 0}$ such that $h(a) \neq h(b)$.

Lemma 3 (Truth lemma)**.** *Let h be a world in the canonical model, and φ a formula. Then $[\![\varphi]\!](h) = (h \circ q)(\varphi)$, where q denotes the natural quotient-map from the set of formulas to $\mathbf{F}$.*

As usual, the proof of the Truth lemma proceeds by induction on φ. The cases for e, $\wedge$, and $\oplus$ are routine, and for $\Box$ the inequality $[\![\Box\varphi]\!](h) \leq (h \circ q)(\Box\varphi)$ follows from the definition of R. For the converse, we need to find a witness h' that is a successor of h such that $(h' \circ q)(\varphi) = (h \circ q)(\Box\varphi)$. The requirement that h' is a successor of h amounts to the requirement that $h(\Box a) \leq h'(a)$ for all a, so we find an infinite system of requirements. Using a topological compactness argument we reduce this to finitely many requirements, so that we can employ Farkas' lemma in order to find the desired h'.

Together, the Separation lemma and Truth lemma imply that $\mathbf{F}_\Box$ embeds into the complex algebra of the canonical model. Hence, our construction can be seen as a representation theorem, albeit only for the free algebra $\mathbf{F}_\Box$. In particular, this shows that any generalised quasi-equation valid in the complex algebra of the canonical model gives rise to an admissible rule in the logic. Restricting attention to equations, the completeness part of the Completeness theorem follows, i.e., $\models_{\mathsf{K}(\mathbf{R}_{\leq 0})} \varphi \approx \psi$ implies $\mathcal{Q}_{\mathrm{msm}} \models \varphi \approx \psi$. For soundness, it is easy to check that any complex algebra satisfies all the quasi-equations in the axiomatisation.

Future work. The results presented here represent a first step in an effort to obtain canonical models for real-valued modal logics, in particular Abelian modal logic and Łukasiewicz modal logic, with a view to establishing completeness results for suitable axiomatizations. A further step towards this goal would be to investigate either the logic considered in this work, or the corresponding fragment of Abelian modal logic studied in [5], extended with the binary join operator $\vee$ (interpreted as max in $\mathbb{R}$). However, it is currently unclear how to adapt the applications of Farkas' lemma in our proofs to deal with joins.

References

[1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic.* Cambridge University Press, 2001.

[2] E. Casari. Comparative logics and abelian ℓ-groups. In C. Bonotto, R. Ferro, S. Valentini, and A. Zanardo, editors, *Logic Colloquium '88*, pages 161–190. Elsevier, 1989.

[3] D. Diaconescu, G. Metcalfe, and L. Schnüriger. A real-valued modal logic. *Logical Methods in Computer Science*, 14(1):1–27, 2018.

[4] G. Hansoul and B. Teheux. Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 101(3):505–545, 2013.

[5] E. Manseau. A fragment of abelian modal logic. Master's thesis, University of Bern, 2024.

[6] R. K. Meyer and J. K. Slaney. Abelian logic from A to Z. In *Paraconsistent Logic: Essays on the Inconsistent*, pages 245–288. Philosophia Verlag, 1989.