Generalization of terms up to equational theory

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A term s is a *generalization* of a term t if t can be obtained from s by variable substitution. The problem of identifying common generalizations for two or more terms has been the focus of substantial research, initiated in a series of papers by Plotkin [6], Popplestone [7], and Reynolds [8], all collected in the same volume published in 1970. The objective of these initial papers was to formalize an abstraction of the process of *inductive reasoning*. The main idea in this context is to find the solutions, i.e., the generalizing terms, that are *as close as possible* to the initial terms that define the problem. The existence and cardinality of this set of "best" solutions is encoded in what is called *generalization type*, which is a main object of study in this topic.

We provide a novel foundational approach to *equational* generalization, i.e., where terms are understood to be equivalent up to an equational theory. The extension to generalization up to equational theories has been considered by several authors in theoretical computer science, and there is a growing interest in a general, foundational approach (see the recent survey [3]). We observe that relevant results so far have been obtained with ad hoc techniques developed for the specific equational theory under consideration (see e.g. results on semirings [1] and idempotent operations [2]).

The collection of methods developed to compute solutions to a generalization problem often goes under the name of *anti-unification*. This terminology suggests a connection between generalization and the arguably better known *unification* problems, where one seeks common instantiations to pairs of given terms. Our approach is indeed inspired by Ghilardi's algebraic setting for the study of equational unification problems [4].

Generally speaking, our methods are those of universal algebra, which is a most natural environment to handle equational theories from the side of their classes of models, i.e., *varieties*. In more detail, we first introduce a purely algebraic representation of equational generalization problems (called *e-generalization problems* from now on) and their solutions; secondly, we develop a universal-algebraic methodology for studying the e-generalization type, applying it in particular to (algebraizable) logics, where the considered equational theory is that of logical equivalence.

We show that e-generalization problems always have a best solutions (i.e., unitary type), in the following varieties: (abelian) groups, (commutative) semigroups and monoids; all varieties whose 1-generated free algebra is trivial, e.g., lattices, semilattices, varieties without constants whose operations are idempotent; Boolean algebras, Kleene algebras and Gödel algebras, which are the equivalent algebraic semantics of, respectively, classical, 3-valued Kleene, and Gödel-Dummett logic.

1 Symbolic e-generalization

The following definition of an e-generalization problem corresponds to the usual one used in the literature, just rephrased in the context of varieties and their free algebras; we only observe that while in the literature e-generalization problems are often considered to be just a pair of terms, we here consider the more general case of allowing a finite set of terms of any cardinality.

Definition 1. *A* symbolic e-generalization problem *for a variety* V *is a finite set* $\mathbf{t}$ *of terms* $t_1, \ldots, t_m \in \mathbf{F}_{\mathsf{V}}(X)$ *for some finite set of variables* X*. A* solution *(or* generalizer*) is a term* $s \in \mathbf{F}_{\mathsf{V}}(Y)$*, with* $Y = \mathrm{Var}(s)$*, for which there exist substitutions* $\sigma_1, \ldots, \sigma_m$ *such that* $\mathsf{V} \models \sigma_k(s) \approx t_k$ *for all* $k = 1, \ldots, m$*. In this case we say that* s *is* witnessed *(or* testified*) by* $\sigma_1, \ldots, \sigma_m$*.*

Any symbolic generalization problem $t_1, \ldots, t_m$ always has a solution: a fresh variable z, testified by the obvious substitutions $\sigma_k(z) = t_k$ for $k = 1, \ldots, m$. This is the *most general solution* for $t_1, \ldots, t_m$, in the sense that every other solution can be obtained from it by further substitution. In this context the interesting solutions are the *least general* ones, that are as *close* as possible to the initial terms representing the problem. Let us make these notions precise.

Consider two terms over the same language s and u; we say that s is *less general* than u, and write

$$s \preceq u, \text{ iff there exists a substitution } \sigma \text{ such that } \sigma(u) = s. \tag{1}$$

Let us then fix a problem $\mathbf{t} \subseteq \mathbf{F}_{\mathsf{V}}(X)$ and let $\mathscr{S}(\mathbf{t})$ be the set of its solutions; $\preceq$ is a preorder on $\mathscr{S}(\mathbf{t})$. With a slight abuse of notation we denote by $(\mathscr{S}(\mathbf{t}), \preceq)$ its associated poset of equally general solutions, that we call the *generality poset* of $\mathbf{t}$. Given a symbolic e-generalization problem $\mathbf{t}$, its *e-generalization type* is either: unitary, finitary, infinitary, or nullary, depending on the cardinality of any minimal (complete) set of solutions in $(\mathscr{S}(\mathbf{t}), \preceq)$.

Given a variety V, its symbolic e-generalization type is the worst possible type occurring among all its e-generalization problems, the best-to-worst order being: unitary > finitary > infinitary > nullary.

2 Algebraic e-generalization

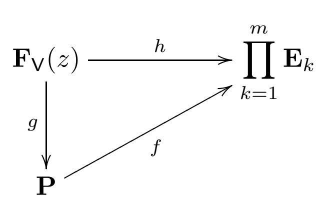
The algebraic translation of e-generalization problems uses projective and exact algebras in the considered variety. Let us call an algebra $\mathbf{A}$ a *retract* of an algebra $\mathbf{F}$ if there are homomorphisms $i : \mathbf{A} \to \mathbf{F}, j : \mathbf{F} \to \mathbf{A}$ such that $j \circ i = id_{\mathbf{A}}$ (and then necessarily i is injective and j is surjective); in this case we say that $\mathbf{A}$ is an (i, j)-retract of $\mathbf{F}$.

Consider a variety V. An algebra $\mathbf{P} \in \mathsf{V}$ is *projective in* V if it is a retract of a free algebra in V; an algebra $\mathbf{E} \in \mathsf{V}$ is called *exact in* V if it is isomorphic to a finitely generated subalgebra of some (finitely generated) free algebra. Evocatively, if we consider an exact algebra that is (isomorphic to) a subalgebra of a free algebra $\mathbf{F}_{\mathsf{V}}(X)$ generated by a term t, we write it as $\mathbf{E}(t)$.

The key idea that makes our translation works is to see the terms $t_1, \ldots, t_m$ representing the problem as a *single element* $(t_1, \ldots, t_m)$ of the direct product of the exact algebras $\mathbf{E}(t_k)$; it will be convenient to represent this tuple as the image of a fresh variable z via some map, which extends to a homomorphism on the 1-generated algebra in the associated variety, $\mathbf{F}_{\mathsf{V}}(z)$.

Definition 2. *We call an* algebraic e-generalization problem *for a variety* V *a homomorphism* $h : \mathbf{F}_{\mathsf{V}}(z) \longrightarrow \prod_{k=1}^{m} \mathbf{E}_k$ *for some* $m \geq 1$*, where each* $\mathbf{E}_k$ *is an exact algebra in* V*.*

A solution *(or* generalizer*) for* h *is any homomorphism* $g : \mathbf{F}_{\mathsf{V}}(z) \longrightarrow \mathbf{P}$*, where* $\mathbf{P}$ *is finitely generated and projective in* V*, for which there exists a homomorphism* $f : \mathbf{P} \longrightarrow \prod_{k=1}^{m} \mathbf{E}_k$ *such that* $f \circ g = h$*, as illustrated in the following diagram:*



We say that f witnesses *or* testifies *the solution* g*.*

Let us define a generality order among algebraic solutions. Fix an algebraic problem $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$. Given two generalizers $g : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}$, $g' : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}'$, we say that g is *less general* than g' and we write

$$g \sqsubseteq g' \text{ if and only if there exists } h : \mathbf{P}' \to \mathbf{P} \text{ such that } h \circ g' = g. \tag{2}$$

The relation $\sqsubseteq$ is easily checked to be a preorder on the set of generalizers for h. We write $(\mathscr{A}(h), \sqsubseteq)$ for the corresponding poset of equally general generalizers. The *algebraic e-generalization type* of a problem is then given in complete analogy with the symbolic case, by checking the cardinality of a minimal complete set of solutions; similarly, one can define the algebraic e-generalization type of a variety as the worst possible type of its problems.

Let us now discuss how to translate back and forth between symbolic and algebraic problems and solutions. Let $\mathbf{t} = \{t_1, \ldots, t_m\} \subseteq \mathbf{F}_{\mathsf{V}}(X)$ be a symbolic e-generalization problem for a variety V, and $s \in \mathbf{F}_{\mathsf{V}}(Y)$ be a solution. Let us define $\mathsf{Alg}(\mathbf{t})$ and $\mathsf{Alg}(s)$ as the (unique) homomorphisms extending the following assignments:

$$\begin{aligned} \mathsf{Alg}(\mathbf{t}) : z \in \mathbf{F}_{\mathsf{V}}(z) \quad &\longmapsto \quad (t_1, \ldots, t_m) \in \prod_{k=1}^{m} \mathbf{E}(t_k), \\ \mathsf{Alg}(s) : z \in \mathbf{F}_{\mathsf{V}}(z) \quad &\longmapsto \quad s \in \mathbf{F}_{\mathsf{V}}(Y). \end{aligned}$$

Conversely, let $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$ be an algebraic e-generalization problem, where each $\mathbf{E}_k$ embeds via a homomorphism e_k to some free algebra $\mathbf{F}_{\mathsf{V}}(X_k)$; consider a solution $g : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}$, where $\mathbf{P}$ is an (i, j)-retract of $\mathbf{F}_{\mathsf{V}}(Y)$. Let p_k be the k-th projection on $\prod_{k=1}^{m} \mathbf{E}_k$, we define:

$$\begin{aligned} \mathsf{Sym}(h) &= \{t_1, \ldots, t_m\}, \text{ where } t_k = e_k \circ p_k \circ h(z) \text{ for } k = 1, \ldots m; \\ \mathsf{Sym}(g) &= (i \circ g)(z) \in \mathbf{F}_{\mathsf{V}}(Y). \end{aligned}$$

We can prove the following.

Theorem 3. *A symbolic e-generalization problem* $\mathbf{t} \subseteq \mathbf{F}_{\mathsf{V}}(X)$ *has a term* $s \in \mathbf{F}_{\mathsf{V}}(Y)$ *as solution if and only if* $\mathsf{Alg}(s)$ *is a solution to* $\mathsf{Alg}(\mathbf{t})$*; conversely, an algebraic e-generalization problem* $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$ *has a homomorphism* $g : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}$ *as a solution if and only if* $\mathsf{Sym}(g)$ *is a solution to* $\mathsf{Sym}(h)$*.*

Corollary 4. *Given a variety* V*, its symbolic and algebraic e-generalization types coincide.*

3 E-generalization type via congruences

After developing the general theory, we take advantage of some basic universal-algebraic tools, and develop a methodology based on the study of the congruence lattice of the 1-generated free algebra in the considered variety. In particular, we identify a class of varieties where the study of the generality type can be fully reduced to the study of this congruence lattice. Let us first transfer the notions of projective and exact from algebras to congruences: we call a congruence θ of a free algebra $\mathbf{F}_{\mathsf{V}}(X)$ *projective* (or *exact*) if $\mathbf{F}_{\mathsf{V}}(X)/\theta$ is projective (or exact) in V.

Definition 5. *We say that an algebra* $\mathbf{S}$ *in a variety* V *is* strongly projective *in* V *if it is projective in* V*, and whenever there is an embedding* $i : \mathbf{S} \to \mathbf{P}$*, for some projective algebra* $\mathbf{P}$*, there is a homomorphism* $j : \mathbf{P} \to \mathbf{S}$ *such that* $j \circ i = id_{\mathbf{S}}$*. We say that a variety* V *is 1ESP if all 1-generated exact algebras in* V *are strongly projective in* V*.*

Given any e-generalization problem h, let us call $\mathscr{G}(h)$ the set of all congruences that appear as kernels of its solutions; in other words, $\mathscr{G}(h)$ is the image under the function ker of the set $\mathscr{A}(h)$, given by all solutions to h: $\mathscr{G}(h) = \ker[\mathscr{A}(h)]$. In varieties such that every 1-generated exact algebra is (strongly) projective, one can show that:

$$\mathscr{G}(h) = \{\theta \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}}(z)) : \theta \subseteq \ker(h), \theta \text{ projective in } \mathsf{V}\}.$$

Theorem 6. *Let V be a 1ESP variety, and consider an algebraic e-generalization problem h. Its poset of solutions $\mathscr{A}(h)$ is dually isomorphic to the poset of congruences in $\mathscr{G}(h)$.*

Using this theorem, one can show that both Boolean and Kleene algebras have unitary e-generalization type, as well as all varieties whose 1-generated free algebra is trivial, e.g., lattices, semilattices, varieties without constants whose operations are idempotent.

Finally, we identify a sufficient condition for a problem, and for a variety, to have unitary e-generalization type.

Theorem 7. *Let $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$ be an algebraic e-generalization problem. If $\ker(h)$ is projective then the e-generalization type of h is unitary.*

Corollary 8. *If V is a variety for which finite intersections of exact congruences of $\mathbf{F}_{\mathsf{V}}(z)$ are projective, then V has unitary e-generalization type.*

As a consequence, one can see that the following varieties have unitary e-generalization type: (abelian) groups, (commutative) semigroups and monoids, Gödel algebras.

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