From compact regular spaces to non-classical negations

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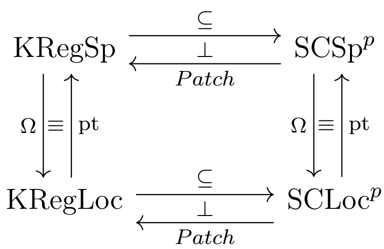
Abstract

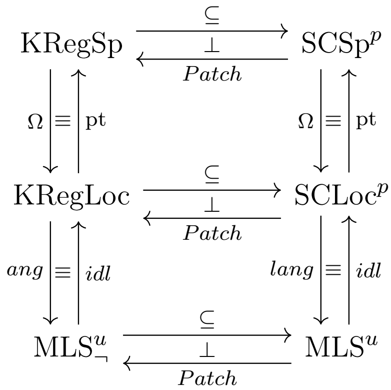
This work investigates 'negation as opposite' and leverages earlier work on compact regularity to provide the necessary proof theory. Specifically, we build on Moshier's connection between Gentzen's cut rule and regularity in compact spaces. Moshier proved that a space is compact regular if and only if every sequent in a corresponding sequent calculus arises as an instance of Gentzen's cut rule. This implies that the logical structure of a generalized version of sequent calculus can fully capture the topological property of compact regularity. Conversely, compact regular spaces can serve as the semantical interpretation of a paraconsistent generalization of sequent calculus [6].

Consider the following diagram



The top of the diagram illustrates a coreflection of stably compact spaces with perfect maps within compact regular spaces with continuous maps. The functors Ω and pt on the sides of the diagram are the adjoints between **Top** and **Loc** derived from Stone duality, reduced to equivalences for the categories in question. The proof that **KRegLoc** and $\mathbf{SCLocp}^p$ consist of spatial locales (and thus Ω and pt reduce to equivalences in these cases) is non-constructive. However, there is a way to give intuitionistically valid proofs: one isolates the intuitionistically valid aspects of the construction in the locales and transfers the results to spaces via Ω and pt. Nevertheless, although localic proofs can be intuitionistically valid, they frequently involve impredicative principles due to the nature of a locale as a type of complete lattice. Quantification over the power set of formal opens of the locale is generally unavoidable, necessitating the sub-object classifier of a topos to discuss locales in general.

Moshier reconsiders the patch construction predicatively, adding another row to the previous diagram. Here, $\mathbf{MLS}^u_\neg$ and $\mathbf{MLS}^u$ are categories equivalent to **KRegLoc** and $\mathbf{SCLoc}^p$. The proofs of these lower equivalences are intuitionistically valid.



MLS, multilingual sequent calculus, was first introduced in [5] and is based on Gentzen's sequent calculus [4]. **MLS** provides finitary formal representations of stably compact locales (or spaces, non-constructively), making it particularly suitable for predicative constructions. The objects in **MLS** are certain sequent relations that are closed under the cut rule and enjoy sequential cut decomposition.

Now, $\mathbf{MLS}^u$ is a full subcategory of **MLS**. It consists of upper adjoint compatible consequence relations. In simpler terms, $\mathbf{MLS}^u$ includes those morphisms in **MLS** that have a corresponding lower adjoint, making them adjoint pairs. This subcategory is significant because it aligns with the category of stably compact locales with perfect maps ($\mathbf{SCLoc}^p$).

The patch construction adds 'opposites' to a given logic. *Opposites* are tokens or formulas that behave according to Gentzen's rules for negation, with the caveat that they are not Boolean complements. Specifically, in a canonically separated calculus, every token ϕ has an opposite ϕ'. Finally, the patch construction ensures that the resulting category ($\mathbf{MLS}^u_\neg$) is equivalent to the category of compact regular locales (**KRegLoc**), hence assuming the prime ideal theorem, to the category of compact Hausdorff spaces.

Opposites are crucial for the patch construction, as they allow the creation of a canonically separated calculus from a given stable calculus. This construction is essential for establishing the coreflection and ensuring that the resulting category has the desired logical and topological properties.

Nevertheless, the author points out that 'earlier versions of the paper used the term "negation", but as an anonymous reviewer and others have pointed out, the reader has a reasonable expectation to interpret negation as a complement', or at least a pseudocomplement. In this talk, we show two things: first, that (philosophically minded) readers do not have a reasonable expectation to interpret negation as a complement. The main reason is that the literature on different types of negations is abundant. For example, some interpret negation as cancellation [7] [9], intuitionists interpret negation as $(A \to \bot)$, negation as incompatibility [2] [8], or that the truth conditions of negation represent the truth within the language [1]; second, as a consequence of the first point, that opposites should be negations of a different sort.

A strong reason to consider opposites as negations lies in their adherence to Gentzen's rules for negation, that is,

$$\frac{\Gamma \vdash \Delta, \phi}{\phi', \Gamma \vdash \Delta}\,\mathrm{L}\sharp \qquad\qquad \frac{\phi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \phi'}\,\mathrm{R}\sharp$$

where ϕ and ϕ' are tokens and if we want to denote they are opposite, we write $\phi \sharp \phi'$.

Whether or not they are Boolean complements depends on the type of negation they represent. While they might align with classical negation, they may also belong to other forms of negation. Indeed, it can be argued that Gentzen's negation rules capture the expected behavior of negation more accurately than Boolean complements. Contrary to the view presented in the original paper, we suggest that opposites can be considered as negations, although they are not Boolean complements. In order to do this some properties different from Boolean complementarity are required to consider a negation as such. The questions is, what are these properties? The literature on negations is well-established in non-classical logics. See [3], [2], [8], [1] for a comprehensive discussion. This perspective allows for the exploration of different types of negations, expanding the scope of opposites to other non-classical negations.

It is worth noticing that neither the law of excluded middle nor its dual, the law of non-contradiction, is satisfied in this framework. Thus, in particular, this logic is classified as paraconsistent. In this talk, we will show that there are other forms of negation that satisfy the properties of opposites as defined in our framework. Opposites, as we have discussed, are tokens that follow Gentzen's rules for negation but are not necessarily Boolean complements. By exploring different types of negations, such as

negation in a paraconsistent setting, or relevant negations, we will show that these forms of negation align with the behavior of opposites. This perspective allows us to expand the scope of opposites to include various non-classical negations, thereby enriching our understanding of negation in logical systems.

Keywords— non-classical negations, Boolean complement, opposites, compact regularity, paraconsistent logic

References

[1] Aron Avron. A non-deterministic view on non-classical negations. *Studia Logica*, 80(64):159–194, 2005.

[2] Jon Michael Dunn. Star and Perp: Two treatments of negation. *Philosophical Perspectives*, 7:331–357, 1993.

[3] Jon Michael Dunn and Chunlai Zhou. Negation in the context of gaggle theory. *Studia Logica*, 80:235–264, 2005.

[4] Gerhard Gentzen. Investigations into logical deduction. In M. Szabo, editor, *The Collected Works of Gerhard Gentzen*, pages 68–131. Amsterdam: North-Holland, Amsterdam, 1969. Originally published as Untersuchungen über das logische Schließen.

[5] Achim Jung, Mathias Kegelmann, and M. Andrew Moshier. Multi lingual sequent calculus and coherent spaces. *Electronic Notes in Theoretical Computer Science*, 6:203–220, 1997. MFPS XIII, Mathematical Foundations of Progamming Semantics, Thirteenth Annual Conference.

[6] M.Andrew Moshier. On the relationship between compact regularity and Gentzen's cut rule. *Theoretical Computer Science*, 316(1):113–136, 2004. Recent Developments in Domain Theory: A collection of papers in honour of Dana S. Scott.

[7] Graham Priest. Negation as cancellation, and connexive logic. *Studia Logica*, 18:141–148, 1999.

[8] Greg Restall. Proof with start and perps. In Shawn Standefer Igor Sedlár and Andrew Tedder, editors, *New Directions in Relevant Logic*, pages 1–27. Springer, forthcoming.

[9] Heinrich Wansing and Daniel Skurt. Negation as cancellation, connexive logic and qlpm. *Australasian Journal of Logic*, 15:476–488, 2018.