The cardinality of intervals of modal and superintuitionistic logics

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1 Introduction

We study the cardinalities of intervals of modal and superintuitionistic logics (si-logics for short). This cardinality cannot be more than the continuum as we assume that our language is countable, and in a countable language we cannot have more than a continuum of logics (which are special sets of formulas). Recall that for modal or si-logics L_1 and L_2 the interval $[L_1, L_2]$ is the set

$$[L_1, L_2] = \{L : L_1 \subseteq L \subseteq L_2\}.$$

These intervals are, clearly, not linearly ordered.

It was first shown by Jankov [6] that there are continuum many si-logics. Therefore, the intervals [IPC, Inconsist] and [IPC, CPC], where IPC and CPC are intuitionistic and classical propositional calculi, respectfully, and Inconsist is the inconsistent logic, have the cardinalities that of the continuum. This was obtained by constructing an antichain of finite subdirectly irreducible Heyitng algebras (alternatively, finite rooted posets) with respect to homomorphic image of a subalgebra order (alternatively, p-morphic image of an upset) and by associating to each such finite algebra the so-called *Jankov formula*, a variant of the diagram of this algebra [6], see also [1]. These results have been generalized to modal logics by Fine [4] and Rautenberg [8] (see [3, Chapter 9] for an overview). For example, the intervals [S4, Inconsist] and [K4, S4] have the cardinality that of the continuum. It is also known that there are some intervals that are finite, e.g., extensions of any tabular transitive modal or si-logic and countably infinite, e.g, the intervals [S4.3, Inconsist] or [LC, Inconsist] (see e.g., [3]).

It was posed as an open problem, only very recently in [5], whether it can be proved without assuming the Continuum Hypothesis (CH) that each interval of modal logics has the cardinality which is countable or that of the continuum. In particular, suppose that for modal or si-logics L_1 and L_2, the interval $[L_1, L_2]$ is not countable, then is the cardinality of this interval that of the continuum, without the use of the CH? This question was triggered by investigations into the degrees of the finite model property (the FMP). This concept was defined in [5] and it was shown that the degree of the FMP of each transitive modal or si-logic can be any finite cardinal, $\aleph_0$ or $2^{\aleph_0}$. With the CH this implies that any cardinality $\leq 2^{\aleph_0}$ can be the degree of the FMP for some transitive modal or si-logic. Because of this, this result was called the Antidichotomy theorem. It was also shown in [5] that the degrees of the FMP for these logics always form an interval.

In this paper, we resolve this open problem affirmatively. We prove this by using techniques from descriptive set theory (see e.g., [7, Section 12]). Specifically, we represent the set of propositional variables as natural numbers and logics as reals. Then sets of logics correspond to some sets of real numbers. We show that for any interval of modal or si-logics the corresponding set of reals is $\mathbf{\Pi}^0_1$, in particular, it is a Borel set. It is a classical result in descriptive set theory that every Borel set has the perfect set property [7]. Thus, the cardinality of such a set is either countable or continuum. As a result, we obtain that every uncountable interval of logics has the cardinality that of the continuum, and the degree of FMP of any transitive modal logic or si-logic can be only any finite cardinal, $\aleph_0$ or $2^{\aleph_0}$ without assuming the CH. This gives the solution to our problem. We also provide a direct proof showing that the degree of FMP in the lattice of normal extensions of any normal modal logic is $\mathbf{\Pi}^0_2$, so the cardinality result also holds for non-transitive modal logics.

As far as we are aware, this perspective on the study of intervals of logics has not been explored before.

2 Main results and proof sketches

We will now move to formal details.

Definition 1. Let L_0 be a normal modal logic.

1. Let $\mathsf{NExt}L_0$ be the lattice consisting of all normal extensions of L_0, namely all normal logics containing L_0, with the order $\subseteq$.

2. Given $L_1 \in \mathsf{NExt}L_0$, let

$$[L_0, L_1] = \{L \text{ normal logic} : L_0 \subseteq L \subseteq L_1\} = \{L \in \mathsf{NExt}L_0 : L \subseteq L_1\}.$$

3. Let FFr be the set of all finite Kripke frames. For $L \in \mathsf{NExt}L_0$, let $\mathsf{FFr}(L) = \{F \in \mathsf{FFr} : F \vDash L\}$, and $\mathsf{fmp}_{L_0}(L) = \{L' \in \mathsf{NExt}L_0 : \mathsf{FFr}(L') = \mathsf{FFr}(L)\}$.

4. The degree of fmp of L in $\mathsf{NExt}L_0$ is the cardinality of the set $\mathsf{fmp}_{L_0}(L)$.

Remark 2. These definitions also apply to si-logics.

Our central observation is that we can identify formulas with natural numbers, logics (which are sets of formulas) with real numbers, and intervals (which are sets of logics) with sets of reals. Since there are countably many proportional variables, we can encode modal formulas in an effective way, that is, $\mathrm{Fml} = \{i \in \omega : i \text{ is a code of a modal formula}\}$ is recursive. ϕ_i will denote the formula with code i. Similarly, given some $A \subseteq \mathrm{Fml}$, let $L_A = \{\phi_i : i \in A\}$. Note that every formula has a unique code and every logic has a unique set of codes. Explicitly, we have $i \in \mathrm{Fml}$ and $\phi_i \in L$ iff $i \in A$, for all $i \in \omega$. We further identify subsets of ω with reals, namely, elements in the Cantor space 2^ω in the canonical way. Under this identification, in particular, logics correspond to elements of 2^ω, and sets of logics to subsets of 2^ω.

This allows us to investigate the arithmetical and Borel complexity of sets of logics, viewed as sets of reals, and apply facts from descriptive set theory (see e.g., [7, Section 12]).

Definition 3. Let $f, g \in 2^\omega$. $f \oplus g \in 2^\omega$ is defined by

$$(f \oplus g)(2n) = f(n) \text{ and } (f \oplus g)(2n+1) = g(n), \text{ for all } n \in \omega.$$

Lemma 4. Let $L_1 \in \mathsf{NExt}L_0$ and $A_1 \subseteq \mathrm{Fml}$ be the code of L_1. Let $[A_0, A_1] = \{A \subseteq \mathrm{Fml} : L_A \in [L_0, L_1]\}$. Then $[A_0, A_1] \in \Pi^0_1(A_0 \oplus A_1)$. Moreover, if L_0 is recursively axiomatizable and L_1 is decidable, then $[A_0, A_1] \in \Pi^0_1$.

Proof Idea. The set $[A_0, A_1] \subseteq 2^\omega$ can be defined by a Π^0_1 formula with parameters A_0, A_1. For example, for $A \subseteq \mathrm{Fml}$, being closed under necessitation is characterized by $\forall i \forall j(\mathrm{Nec}(i,j) \wedge j \in A \to i \in A)$, where $\mathrm{Nec}(x,y)$ is a recursive relation such that $\mathrm{Nec}(i,j)$ iff $i, j \in \mathrm{Fml}$ and ϕ_i is of the form $\Box\phi_j$; being an extension of A_0 is characterized by $\forall i(i \in A_0 \to i \in A)$.

If L_0 is recursively axiomatizable, then A_0 is recursively enumerable, so there is some recursive R such that $A_0 = \{i \in \omega : \exists j R(i,j)\}$. Then, in the defining formula, $i \in A_0$ can be replaced by $\exists j R(i,j)$. Similarly, if L_1 is decidable, $i \in A_1$ can be replaced by a recursive predicate $R'(i)$. These changes keep the formula Π^0_1 and eliminate the parameters. $\Box$

Thus, for any interval of modal or si-logics, the corresponding set of reals is $\mathbf{\Pi}^0_1$, in particular, it is a Borel set. It is a classical result in descriptive set theory that every Borel set has the perfect set property [7]. It follows that the cardinality of such a set is either countable or that of the continuum. As a result, we obtain the main theorem.

Theorem 5. Let $L_1 \in \mathsf{NExt}L_0$ and $A_1 \subseteq \mathrm{Fml}$ be the code of L_1. Then $[L_0, L_1]$ has the cardinality either countable or that of the continuum.

The theorem applies to si-logics with straightforwardly adjusted proofs. The next corollary follows from the fact that $\mathsf{fmp}_{L_0}(L)$ always form an interval in transitive modal logics or si-logics, which was shown in [5]. This gives the solution to our problem.

Corollary 6.

1. Let L_0 be a transitive modal logic, i.e., a normal modal logic containing **K4**. Let $L \in \mathsf{NExt}L_0$. Then $\mathsf{fmp}_{L_0}(L)$ has the cardinality either countable or that of the continuum.

2. Let L be a si-logic. Then $\mathsf{fmp}(L)$ (in the lattice of si-logics) has the cardinality either countable or that of the continuum.

In addition, we generalize the result to non-transitive modal logic L_0 by directly characterizing the complexity of $\mathsf{fmp}_{L_0}(L)$.

Lemma 7. Let $L \in \mathsf{NExt}L_0$ with code A. Let $\mathsf{fmp}_{A_0}(A) = \{A' \subseteq \mathrm{Fml} : L_{A'} \in \mathsf{NExt}L_0, \mathsf{FFr}(L_{A'}) = \mathsf{FFr}(L)\}$. Then $\mathsf{fmp}_{A_0}(A) \in \Pi^0_2(A_0 \oplus A)$. Moreover, if L_0 and L are recursively axiomatizable, then $\mathsf{fmp}_{A_0}(A) \in \Pi^0_2$.

Proof Sketch. A finite Kripke frame is a finite set with a binary relation. So, finite Kripke frames (up to isomorphism) can be coded by natural numbers in an effective way, such that:

1. The set $\mathrm{FFr} = \{f \in \omega : f$ is a code of a finite Kripke frame$\}$ is recursive.

2. The validity relation $\mathrm{Val}(f,i)$ iff f is the code of a finite Kripke frame F and i is the code of a formula ϕ and $F \vDash \phi$ is recursive.

This enables us to define the set $\mathsf{fmp}_{A_0}(A) \subseteq 2^\omega$ by a Π^0_2 formula with parameters A_0, A.

The second half of the statement follows by a similar argument in the proof of Lemma 4. $\Box$

A similar application of the perfect set property of Borel sets gives the next theorem.

Theorem 8. Let L_0 be a normal modal logic. For any $L \in \mathsf{NExt}L_0$ the set $\mathsf{fmp}_{L_0}(L)$ has the cardinality either countable or that of the continuum.

It is worth noting that these proofs do not use any special properties of modal or si-logics. Thus, this approach can be employed to investigate cardinalities of sets of other non-classical logical systems (with a reasonably simple syntax and semantics).

However, not all properties allow such straightforward characterization. A notable example is the degree of Kripke incompleteness. Although Blok [2] proved that the degree of Kripke incompleteness in $\mathsf{NExt}\mathbf{K}$ is either 1 or $2^{\aleph_0}$, the situation in other lattices, such as $\mathsf{NExt}\mathbf{K4}$ and $\mathsf{NExt}\mathbf{S4}$, remains unknown [3, Problem 10.5]. Given that all Kripke frames form a proper class, it becomes challenging to reason about Kripke frames using quantifiers over natural numbers or even reals, in contrast to finite Kripke frames.

We leave open the question of what implications the characterization within the Borel hierarchy may have for studying logical properties beyond the cardinality argument we presented. For example, if a logical property P is shown to be Borel, analytic, or belongs precisely to some complexity class C, what conclusions can we draw about that property in relation to logics?

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