The structure and theory of McCarthy algebras

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In his seminal paper [5], in regards to the theory of computation, John McCarthy introduced a logic for computable functions with the aim of managing *undefined* assignments, partial predicates, and modeling computational failures. As the order in which programs are executed may be paramount, the conjunction/disjunction with an undefined value may fail to commute, and thus yields a *non-commutative* logic. This paradigm has also found application in the study of Process Algebras, such as the handling and management of *errors* in concurrent programming; for instance in [1] where the operation $\cdot$ in Figure 1 is used for *left sequential conjunction.*

The first algebraic treatment for a 3-valued semantics of McCarthy's logic was carried out by Konikowska in [4], where the following operation tables over a set $M_3 := \{0, 1, \varepsilon\}$ are introduced.

$'$	
1	0
0	1
ε	ε

$+$	1	0	ε
1	1	1	1
0	1	0	ε
ε	ε	ε	ε

$\cdot$	1	0	ε
1	1	0	ε
0	0	0	0
ε	ε	ε	ε

Figure 1: The operation tables for the algebra $\mathbf{M}_3 := \langle \{0, 1, \varepsilon\}, +, \cdot, ', 0, 1\rangle$.

As Konikowska defines in [4], an algebra $\langle A, +, \cdot, ', 0, 1\rangle$ is called a **McCarthy algebra** if it *"satisfies all the equational tautologies of a Boolean algebra that hold in"* the algebra $\mathbf{M}_3$. From the observation that the two-element Boolean algebra $\mathbf{2}$ is a subalgebra of $\mathbf{M}_3$, we may restate this, within the parlance of universal algebra, and understand a McCarthy algebra to be any member in the *variety* of algebras generated by $\mathbf{M}_3$. In this way, let us define M to be the **variety of McCarthy algebras** denoting $\mathsf{V}(\mathbf{M}_3)$.

The following properties are readily verified for the algebra $\mathbf{M}_3$, and thus also M:

- the operation $'$ is an *involution*, i.e., $x'' \approx x$, through which the constants $0 \approx 1'$ and $1 \approx 0'$ are inter-definable;
- the operations $+$ and $\cdot$ the term-definable from each other through $'$ via $x + y \approx (x' \cdot y')'$ and $x \cdot y \approx (x' + y')'$, i.e., they satisfy the De Morgan laws;
- the reduct $\langle M_3, \cdot, 1\rangle$ (thus also $\langle M_3, +, 0\rangle$) is a monoid with an *idempotent* operation, i.e., $x \cdot x \approx x$ (thus also $x + x \approx x$).

Let us call an algebra $\langle A, \cdot, ', 1\rangle$ an *unital band with involution* (**i-uband** for short) if $\langle A, \cdot, 1\rangle$ is a unital band (i.e., idempotent monoid) and $'$ an involution on A; we write $\langle A, +, \cdot, ', 0, 1\rangle$ to indicate its term-definable De Morgan dual $\langle A, +, ', 0\rangle$ in the signature.

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Theorem 1. *There are exactly ten non-isomorphic i-ubands of cardinality 3, exactly four of which containing* **2** *as a Boolean subalgebra; the Strong Kleene algebra* **SK***, the Weak Kleene algebra* **WK***, the McCarthy algebra* $\mathbf{M}_3$ *and its mirror* $\mathbf{M}_3^{\mathsf{op}}$ *(i.e., where* $x \cdot^{\mathsf{op}} y := y \cdot x$*).*

While a great deal is known about the Strong and Weak Kleene algebras and the varieties they generate (see e.g. [2, 3, 6]), little is known about the variety M of McCarthy algebras. In the same article [4], Konikowska gives a long list of equational identities that are valid for M, but whether this list forms a complete axiomatization is left open as conjecture. Part of this research settles this question by both demonstrating that Konikowska's identities are indeed complete for M, and also providing a number of equivalent and minimal axiomatizations. We motivate one such presentation as follows.

For one, the algebra $\mathbf{M}_3$ satisfies distributivity from the left:

$$x \cdot (y + x) \approx xy + xz \quad \text{(or, equivalently)} \quad x + yz \approx (x + y) \cdot (x + z) \qquad \text{(left-distributivity)}$$

However, $\langle M_3, +, \cdot \rangle$ is not a semiring as distributivity from the right fails in general. But some instances of this law do hold, in particular the following:

$$(x + x') \cdot y \approx xy + x'y \quad \text{(or, equivalently)} \quad xx' + y \approx (x + y) \cdot (x' + y) \qquad \text{(ortho-distributivity)}$$

Of course, the most glaring identity that fails in $\mathbf{M}_3$ is that of commutativity. Thus the monoid reduct fails to form a semi-lattice. Even worse, $\langle M_3, +, \cdot \rangle$ is not even a skew-lattice, as the right-absorption laws are falsified (e.g., $1 \neq (\varepsilon + 1) \cdot 1 = \varepsilon$). However, $\mathbf{M}_3$ does satisfy the following left-absorption law:

$$x \cdot (x + y) \approx x \quad \text{(or, equivalently)} \quad x + xy \approx x \qquad \text{(left-absorption)}$$

While $\mathbf{M}_3$ is not ortho-complemented, i.e., the identity $1 \approx x + x'$ (equivalently, $0 \approx x \cdot x'$) fails, it does satisfy a *local* version with unary term-operations $0_x := x \cdot 0$ and $1_x := x + 1$:

$$1_x \approx x + x' \quad \text{(or, equivalently)} \quad 0_x \approx x \cdot x' \qquad \text{(locally complemented)}$$

Lastly, while commutativity generally fails, it does satisfy some instances. In particular for the *local units* $1_x := x + 1$ and $0_x := x \cdot 0$:

$$1_x \cdot 1_y \approx 1_x \cdot 1_y \quad \text{(or, equivalently)} \quad 0_x + 0_y \approx 0_x + 0_y \qquad \text{(local-unit commutativity)}$$

Definition 2. We call a *McCarthy-Konikowska algebra* (**MK-algebra**) any i-uband satisfying left-distributivity, ortho-distributivity, left-absorption, locally complemented, and local-unit commutativity. Denote the variety of MK-algebras by MK.

With a good deal of work, we verify the following:

Theorem 3. *Konikowska's axioms [4, (A1–A16) pp. 169] hold in* MK*.*

Among these identities sits that of *left-regularity*, i.e., $xyx \approx xy$. In fact, and while the derivation is far from trivial, any left-distributive i-uband satisfying local-unit commutativity is also left-regular. As is well-known, any left-regular operation $*$ admits a partial order $\leq_*$ defined via $x \leq_* y$ iff $x * y = y$. For MK-algebras, we choose to work with the partial order associated with the operation $+$, and will denote it simply by $\leq$. This fact affords us the following structure theorem for MK-algebras. First, recall the standard notation $\uparrow a := \{x \in A : a \leq x\}$ and $\downarrow b := \{x \in A : x \leq b\}$, and that of an interval $[a, b] := \uparrow a \cap \downarrow b$.

Theorem 4. *Let* $\mathbf{A} = \langle A, +, \cdot, ', 0, 1\rangle$ *be an MK-algebra. Define* $\mathcal{I}_{\mathbf{A}} := \{0_a : a \in A\}$ *and, for each* $i \in \mathcal{I}_{\mathbf{A}}$*, set* $B_i := [0_i, 1_i]$*, where* $0_x := x \cdot 0$ *and* $1_x := x + 1$*. Then the following hold:*

1. $\langle \mathcal{I}_{\mathbf{A}}, \vee, 0\rangle$ *is a join-semilattice with least element* 0*, where* $i \vee j := i + j$*.*
2. *For each* $i \in \mathcal{I}_{\mathbf{A}}$*,* $\mathbf{A}_i := \langle \uparrow 0_i, +, \cdot, ', 0_i, 1_i\rangle$ *is an MK-algebra and the map* $h_i : x \mapsto 0_i + x$ *is a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}_i$*.*
3. *For each* $i \in \mathcal{I}_{\mathbf{A}}$*, the structure* $\mathbf{B}_i := \langle B_i, +, \cdot, ', 0_i, 1_i\rangle$ *is a Boolean algebra and the set* B_i *coincides with* $\{x \in A : 0_x = 0_i\}$*. Consequently,* $A = \bigcup_{i\in\mathcal{I}_{\mathbf{A}}} B_i$ *and the members of* $\{B_i\}_{i\in\mathcal{I}_{\mathbf{A}}}$ *are pairwise disjoint.*
4. *For each* $i, j \in \mathcal{I}_{\mathbf{A}}$ *with* $i \leq j$*, the map* $\rho_{ij} := h_i \restriction B_i$ *is a homomorphism from* $\mathbf{B}_i$ *to* $\mathbf{B}_j$*. Moreover,* $\rho_{ii} = \mathrm{id}_{\mathbf{B}_i}$ *and* $\rho_{jk} \circ \rho_{ik} = \rho_{ik}$ *for each* $i \leq j \leq k$ *in* $\mathcal{I}_{\mathbf{A}}$*.*

This structure theorem allows for a finer analysis of MK-algebras, in particular those that are subdirectly irreducible, and ultimately serves as the linchpin for the following characterization.

Theorem 5. *The only subdirectly irreducible MK-algebras are the two-element Boolean algebra* $\mathbf{2}$ *and the* 3*-element MK-algebra* $\mathbf{M}_3$*.*

As every variety of algebras is generated by its subdirectly irreducible members, and $\mathbf{2}$ is a subalgebra of $\mathbf{M}_3$, we immediately obtain the following as a corollary to Theorem 5.

Corollary 6. *The variety of MK-algebras is generated by the algebra* $\mathbf{M}_3$*. Consequently, the variety of McCarthy algebras coincides with* MK*.*

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