# Hölder's theorem for totally ordered monoids

Adam Přenosil[1] and Luca Spada[2]

[1] Universitat de Barcelona
adam.prenosil@gmail.com
[2] Università degli Studi di Salerno
lspada@unisa.it

Hölder's theorem [3, 5, 6], one of the early classical results about ordered groups, states that a totally ordered group embeds into the additive ordered group of reals $\mathbb{R}$ if and only if it is Archimedean (informally speaking, if and only if it lacks infinitesimal elements). The ordered group $\mathbb{R}$ which features in Hölder's theorem lives in the variety of lattice-ordered groups, which is one of the most prominent varieties of residuated lattices. A fruitful research programme in this area has been to extend results about lattice-ordered groups to wider classes of residuated lattices [1]. Two important classes for this purpose are the variety of GBL-algebras and its subvariety of GMV-algebras [2]. These significantly extend the variety of lattice-ordered groups while still preserving some group-like behavior. In particular, in their study of the Archimedean property in residuated lattices, Ledda, Paoli and Tsinakis [4] recently extended Hölder's theorem to GBL-algebras, characterizing the subalgebras of the GMV-algebras $\mathbb{R}$, $\mathbb{R}^-$, and $[0, 1]$ as precisely the strongly simple GBL-algebras (see below for more details). In the present work, we further extend Hölder's theorem beyond the residuated setting, obtaining a result in the spirit of [4] for totally ordered monoids which gives an abstract characterization of the *dense* subalgebras of the totally ordered monoids $\mathbb{R}$, $\mathbb{R}^-$, and $[0, 1]$.

Some definitions will be needed to state these results. A *lattice-ordered monoid*, or *$\ell$-monoid* for short, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, 1\rangle$ which is both a lattice and a monoid such that products distribute over binary meets and binary joins. An $\ell$-monoid is *integral* if the monoidal unit 1 is the top element of the lattice reduct. A *totally ordered monoid*, or *tomonoid* for short, is an $\ell$-monoid whose lattice reduct is totally ordered. A *residuated lattice* is an $\ell$-monoid equipped with binary operations $\backslash$ and $/$ such that

$$y \leq x\backslash z \iff x \cdot y \leq z \iff x \leq z/y.$$

A *GBL-algebra* is a residuated lattice satisfying the divisibility equations

$$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x.$$

A *GMV-algebra* is a residuated lattice satisfying the stronger equations

$$x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x.$$

The variety of GMV-algebras in particular subsumes the varieties of $\ell$-groups and MV-algebras. Key examples of commutative GMV-algebras are the additive $\ell$-group of the reals $\mathbb{R}$, its negative cone $\mathbb{R}^-$ (an integral cancellative residuated lattice), and the standard MV-chain $[0, 1]$. These GMV-algebras have some important subalgebras: the additive $\ell$-group of the integers $\mathbb{Z}$, its negative cone $\mathbb{Z}^-$, and the subalgebras $Ł_n$ of $[0, 1]$ with the universes $\{0/n, 1/n, \dots, n/n\}$ for $n \geq 1$. We use $Ł_0$ to denote the trivial algebra.

A residuated lattice is *strongly simple* if it has no non-trivial proper convex subalgebras. It is *strongly semisimple* if $\{1\}$ is the intersection of all maximal proper convex subalgebras. A commutative residuated lattice is strongly (semi)simple if and only if it is (semi)simple in the universal algebraic sense.

**Hölder's theorem for GBL-algebras** ([4, Theorem 5.6])**.** *A GBL-algebra is strongly simple if and only if it is isomorphic to one of the following:*

*(i) a subalgebra of* $\mathbb{R}$*,*

*(ii) a subalgebra of* $\mathbb{R}^{-}$*,*

*(iii) a subalgebra of* $[0,1]$*.*

*In particular, each strongly simple GBL-algebra is commutative.*

We now aim to extend this theorem to totally ordered monoids which are not necessarily residuated. This will require two modifications.

Firstly, the lattice of convex subalgebra is really a stand-in for the lattice of left congruences, or equivalently for the lattice of right congruences. A *left congruence* of an $\ell$-monoid $\mathbf{L}$ is a lattice congruence $\theta$ such that $\langle a,b\rangle \in \theta$ implies $\langle ca, cb\rangle \in \theta$, and a *right congruence* is a lattice congruence $\theta$ such that $\langle a,b\rangle \in \theta$ implies $\langle ac, bc\rangle \in \theta$. A left congruence of a residuated lattice moreover satisfies the condition that $\langle a,b\rangle \in \theta$ implies $\langle c\backslash a, c\backslash b\rangle \in \theta$, while a right congruence satisfies the condition that $\langle a,b\rangle \in \theta$ implies $\langle a/c, b/c\rangle \in \theta$. In a residuated lattice, the lattices of left congruences, of right congruences, and of convex subalgebras are isomorphic. Beyond the residuated case, we need to explicitly work with the lattices of left and right congruences.

Secondly, as a result of dropping residuation from the signature, we now have more (left and right) congruences. The strongly simple residuated lattices $\mathbb{R}$, $\mathbb{R}^{-}$, and $[0,1]$ have no residuated lattice congruences besides the identity and the total congruence. In contrast, for each non-empty downset $I$ of $[0,1]$ there is an $\ell$-monoidal congruence $\Theta(I)$ such that $\langle a,b\rangle \in \Theta(I)$ if and only if either $a=b$ or $a,b \in I$. The same holds for $\mathbb{R}^{-}$. The characteristic condition is no longer that there are no congruences besides the identity and the total congruence. Rather, it is that every congruence arises has the form $\Theta(I)$.

More generally, given an $\ell$-monoid $\mathbf{L}$, a *left (right) ideal* of $\mathbf{L}$ is a non-empty downset $I$ which is both a lattice ideal ($a,b \in I$ implies $a \vee b \in I$) and a left (right) ideal of the monoid reduct: if $i \in I$ and $a \in \mathbf{L}$, then $a \cdot i \in \mathbf{L}$ ($i \cdot a \in \mathbf{L}$). Each left (right) ideal $I$ of $\mathbf{L}$ induces a left (right) congruence of $\mathbf{L}$ as follows:

$$\langle a,b\rangle \in \Theta(I) \iff a \vee i = b \vee i \text{ for some } i \in I.$$

Such congruences will be called *left (right) ideal congruences*. Notice that in an integral tomonoid the left (right) ideals are precisely the downsets, and that each left (right) ideal congruence of a tomonoid has at most one non-trivial congruence class and it is a downset.

An *ideal $\ell$-monoid* is an $\ell$-monoid where each non-identity left congruence is a left ideal congruence and each non-identity right congruence is a right ideal congruence, excluding the pathological case of $\ell$-monoids isomorphic to the two-element additive tomonoid $\{0,+\infty\}$. The following theorem gives a concrete description of ideal tomonoids – or conversely, an abstract characterization of the dense subtomonoids of $\mathbb{R}$, $\mathbb{R}^{-}$, and $[0,1]$, just like the previous theorem gave an abstract characterization of the residuated subtomonoids of $\mathbb{R}$, $\mathbb{R}^{-}$, and $[0,1]$.

**Hölder's theorem for ideal tomonoids.** *A tomonoid is an ideal tomonoid if and only if it is isomorphic to one of the following:*

*(i)* $\mathbb{Z}$*,*

*(ii)* $\mathbb{Z}^{-}$*,*

*(iii)* $L_n$ *for some* $n \in \mathbb{N}$*,*

*(iv)* *a dense subtomonoid of* $\mathbb{R}$,

*(v)* *a dense subtomonoid of* $\mathbb{R}^-$,

*(vi)* *a dense subtomonoid of* $[0, 1]$.

*In particular, each ideal tomonoid is commutative.*

The restriction to dense subtomonoids which occurs of the above theorem was already implicit in Hölder's theorem for GBL-algebras: every subgroup of $\mathbb{R}$ and more generally every residuated sublattice of $\mathbb{R}$, $\mathbb{R}^-$, and $[0, 1]$ is either dense or isomorphic to $\mathbb{Z}$, $\mathbb{Z}^-$, or $Ł_n$ for some $n \in \mathbb{N}$. In contrast, these algebras have subtomonoids which are neither dense nor isomorphic to $\mathbb{Z}$, $\mathbb{Z}^-$, or $Ł_n$, and which therefore fail to be ideal tomonoids.

To better understand the role of density, consider the subtomonoid $[0, 1/2] \cup \{1\}$ of $[0, 1]$. This is a tomonoid equipped with a drastic multiplication: $1 \odot x = x = x \odot 1$, otherwise $x \odot y = 0$. In this tomonoid, the principal congruence $\langle 1/4, 1/2 \rangle$ is not an ideal congruence, since the equivalence class of $1/2$ is the non-singleton interval $[1/4, 1/2]$, which is not a downset. However, had the tomonoid contained for instance the element $9/10$, its presence would force the equivalence class of $1/2$ to be the interval $[0, 1/2]$ rather than $[1/4, 1/2]$.

The problem of describing all ideal $\ell$-monoids, as opposed to merely ideal tomonoids, remains open. We can, however, describe the finite ideal $\ell$-monoids. These coincide with the finite semisimple GMV-algebras, or in other words with finite MV-algebras.

The following result was first formulated as a conjecture by Peter Jipsen. This conjecture is where the condition of being an ideal $\ell$-monoid (more precisely, an ideal join-semilattice-ordered commutative monoid) was first isolated.

**Theorem.** *Finite ideal $\ell$-monoids are precisely the $\ell$-monoid reducts of finite MV-algebras, i.e. up to isomorphism they are the finite products of the $\ell$-monoids $Ł_n$ for $n \in \mathbb{N}$.*

In particular, all finite ideal $\ell$-monoids are reducts of finite GMV-algebras. This does not hold beyond the finite case. However, every ideal $\ell$-monoid does satisfy a natural $\ell$-monoidal version of the GMV property, namely the last condition in the following equivalence.

**Fact.** *The following are equivalent for every residuated lattice* $\mathbf{L}$*:*

*(i)* $\mathbf{L}$ *is a GMV-algebra, i.e. it satisfies the following equations:*

$$x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x.$$

*(ii)* $\mathbf{L}$ *satisfies the following implications for* $z \leq x \leq y$*:*

$$x\backslash z \leq y\backslash z \implies y \leq x, \qquad\qquad z/x \leq z/y \implies y \leq x.$$

*(iii)* $\mathbf{L}$ *satisfies the following implications for* $z \leq x \leq y$*:*

$$(xu \leq z \text{ implies } yu \leq z) \text{ for all } u \in \mathbf{L} \implies y \leq x,$$
$$(ux \leq z \text{ implies } uy \leq z) \text{ for all } u \in \mathbf{L} \implies y \leq x.$$

# References


[1] Michal Botur, Jan Kühr, Lianzhen Liu, and Constantine Tsinakis. The Conrad program: from $\ell$-groups to algebras of logic. *Journal of Algebra*, 450:173–203, 2016.

[2] Nikolaos Galatos and Constantine Tsinakis. Generalized MV-algebras. *Journal of Algebra*, 283(1):254–291, 2005.

[3] Otto Hölder. Die Axiome der Quantität und die Lehre vom Mass. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikaliche Classe*, 53:1–64, 1901.

[4] Antonio Ledda, Francesco Paoli, and Constantine Tsinakis. The Archimedean property: new horizons and perspectives. *Algebra universalis*, 79:1–30, 2018.

[5] Joel Mitchell and Catherine Ernst. The axioms of quantity and the theory of measurement: translated from part I of Otto Hölder's German text "Die Axiome der Quantität und die Lehre vom Mass". *Journal of Mathematical Psychology*, 40(3):235–252, 1996.

[6] Joel Mitchell and Catherine Ernst. The axioms of quantity and the theory of measurement: translated from part II of Otto Hölder's German text "Die Axiome der Quantität und die Lehre vom Mass". *Journal of Mathematical Psychology*, 41(4):345–356, 1997.