Axiomatizing Small Varieties of Periodic ℓ-pregroups

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A *lattice-ordered pregroup* (ℓ-pregroup) is an algebra $(L, \wedge, \vee, \cdot, {}^\ell, {}^r, 1)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leq$ and for every $a \in L$,

$$a^\ell a \leq 1 \leq aa^\ell \text{ and } aa^r \leq 1 \leq a^r a.$$

Lattice-ordered pregroups can be seen as a generalization of lattice-ordered groups (ℓ-groups) which have been extensively studied [1, 4, 9]. Indeed, ℓ-groups correspond exactly to the ℓ-pregroups that satisfy $x^\ell \approx x^r$ and in this case x^ℓ is the group inverse operation. On the other hand ℓ-pregroups are a special case of *pregroups* defined similarly to ℓ-pregroups but without demanding that its underlying order is a lattice. Pregroups where introduced in the context of mathematical linguistics [2, 3, 10]. Moreover, ℓ-pregroups are exactly the residuated lattices that satisfy $(xy)^\ell \approx y^\ell x^\ell$ and $x^{r\ell} \approx x \approx x^{\ell r}$, where $x^\ell := x\backslash 1$ and $x^r := 1/x$. So the methods developed for residuated lattices and their connection to substructural logics (see e.g., [8]) also apply to ℓ-pregroups.

An ℓ-pregroup is called *distributive* if its lattice reduct is distributive. The variety DLP of distributive ℓ-pregroups was studied in depth in [5] where a Holland-style representation theorem is obtained and shown that DLP has a decidable equational theory.

In this work we will restrict ourselves to periodic ℓ-pregroups. An ℓ-pregroup is called *n-periodic* for $n \in \mathbb{N}$ if it satisfies the equation $x^{\ell^n} \approx x^{r^n}$. As noted above, 1-preiodic ℓ-pregroups correspond exactly to ℓ-groups. We denote the variety of n-periodic ℓ-pregroups by LP_n. In [7] it was shown that every periodic ℓ-pregroup is distributive. Moreover, in [6] a representation theorem for periodic ℓ-pregroups is obtained and it is shown that the equational theory of LP_n is decidable for each $n \in \mathbb{N}$.

Let $f\colon \mathbf{P} \to \mathbf{Q}$ and $g\colon \mathbf{Q} \to \mathbf{P}$ be maps between posets. We say that g is a *residual* for f and f is a dual residual for g if for all $p \in P$, $q \in Q$,

$$f(p) \leq q \iff p \leq g(q).$$

The residual and dual residual of a map f are unique if they exist and we denote them by f^r and f^ℓ, respectively. Inductively, we define the nth-order residual if it exists, by $f^{r^1} = f^r$ and $f^{r^{n+1}} = (f^{r^n})^r$ and analogously we define the nth-order dual residual of f.

For a chain $\mathbf{\Omega}$ we denote by $F(\mathbf{\Omega})$ the set of maps on $\mathbf{\Omega}$ that have residuals and dual residuals of every order. This set gives rise to a distributive ℓ-pregroup $\mathbf{F}(\mathbf{\Omega}) = (F(\mathbf{\Omega}), \wedge, \vee, \circ, {}^\ell, {}^r, id_\Omega)$, where $\wedge$ and $\vee$ are defined point-wise, $\circ$ is functional composition, and id_Ω is the identity map on Ω. In [5] it was shown that $\mathbf{F}(\mathbb{Z})$ generates DLP. The subset $F_n(\mathbf{\Omega})$ of $\mathbf{F}(\mathbf{\Omega})$ of the maps that satisfy $f^{r^\ell} = f^{r^n}$ forms an n-periodic subalgebra $\mathbf{F}_n(\mathbf{\Omega})$ of $\mathbf{F}(\mathbf{\Omega})$. In contrast to the result about DLP it was shown in [6] that $\mathbf{F}_n(\mathbb{Z})$ does not generate LP_n for any $n \in \mathbb{N}$, but LP_n is generated by $\mathbf{F}_n(\mathbb{Q}\overrightarrow{\times}\mathbb{Z})$ for every $n \in \mathbb{N}$. Nevertheless it was shown that the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is decidable and that $\bigvee_{n\in\mathbb{N}} \mathsf{LP}_n = \bigvee_{n\in\mathbb{N}} \mathbb{V}(\mathbf{F}_n(\mathbb{Z})) = \mathsf{DLP}$, yielding two different ways of approximating DLP with varieties of periodic ℓ-pregroups. A problem left open in [6] is whether the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is finitely axiomatizable. In this work we show that $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized relative to LP_n by a single equation. Let us define $x^{[k]} = x^{\ell^{2k}}$ and $\sigma_n = x \wedge x^{[1]} \wedge \ldots \wedge x^{[n-1]}$. For an n-periodic ℓ-pregroup $\mathbf{L}$ and $a \in L$ the element $\sigma_n(a)$ is exactly the minimal invertible element above a. Our main result can now be stated:

Theorem. *For each n the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized relative to LP_n by the equation $x\sigma_n(y)^n \approx \sigma_n(y)^n x$.*

The theorem is proved in three steps. First we connect the congruence lattice of n-periodic ℓ-pregroups to the congruence lattice of their subalgebra of invertible elements which we call the *group skeleton*. In fact the group skeleton is exactly the image of the term operation σ_n. Then, using a decomposition theorem for periodic ℓ-pregroup of [6], we characterize the finitely generated subdirectly irreducible n-periodic ℓ-pregroups that satisfy $x\sigma_n(y)^n \approx \sigma_n(y)^n x$ as lexicographic products of a totally ordered abelian group and $\mathbf{F}_k(\mathbb{Z})$, where k devides n. Finally we show that all of these finitely generated subdirectly irreducibles are contained in $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$. In particular, on the way we obtain the following characterizations of (finitely) subdirectly irreducible members of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$.

Corollary. *The finitely generated subdirectly irreducible members of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ are exactly the lexicographic producs of a finitely generated totally ordered abelian group and $\mathbf{F}_k(\mathbb{Z})$ for some k that devides n.*

Corollary. *The finitely subdirectly irreducible members of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ are exactly the n-periodic ℓ-pregroups whose group skeleton is a totally ordered abelian group.*

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