Extending twist construction for Modal Nelson lattices

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This work is about one of the most challenging trends of research in non-classical logic which is the attempt to combine different non-classical approaches together, in our case many-valued and modal logic. This kind of combination offers the skill of dealing with modal notions like belief, knowledge, and obligations, in interaction with other aspects of reasoning that can be best handled using many-valued logics, for instance, vagueness, incompleteness, and uncertainty. In fact, the study that we are going to introduce could be especially interesting from the point of view of Theoretical Computer Science and Artificial Intelligence.
In the present study, we consider the extension of Nelson residuated lattices (N3) with an unary modal operator. We introduce a variety of modal Nelson lattices which we prove that they are characterized by twist structures.

In order to reach this result, we will first introduce an extension for the modal setting of the one well-known construction of Nelson lattices called twist structures, whose importance has been growing in recent years within the study of algebras related to non-classical logics (see [1, 2, 5]). Our proposed extension is more general than others considered in the literature because it is not required to be monotone with respect to modal operators (see [4]).

We assume the reader knows the main properties and definitions about residuated lattices and Heyting algebras. In addition, a residuated lattice is called involutive if it is bounded and it satisfies the double negation equation:

$$a = \neg\neg a.$$

A Nelson residuated lattice or simply Nelson lattice (N3) is an involutive residuated lattice satisfying:

$$((a^2 \to b) \wedge ((\neg b)^2 \to \neg a)) \to (a \to b) = \top.$$

Definition 1. Given a Heyting algebra $\mathbf{H}$, we shall denote by $D(\mathbf{H})$ the filter of dense elements of $\mathbf{H}$, i.e. $D(\mathbf{H}) = \{x \in H : -x = \bot\}$.

A filter F of $\mathbf{H}$ is said to be Boolean provided the quotient $\mathbf{H}/F$ is a Boolean algebra. It is well known and easy to check that a filter F of the Heyting algebra $\mathbf{H}$ is Boolean if and only if $D(\mathbf{H}) \subseteq F$. The Boolean filters of $\mathbf{H}$, ordered by inclusion, form a lattice, having the improper filter H as the greatest element and $D(\mathbf{H})$ as the smallest element.
With all these elements, we can reproduce the twist-structures corresponding to N3-lattices.

Theorem 2. *(Sendlewski + Theorem 3.1 in [1].) Given a Heyting algebra*

$$\mathbf{H} = \langle H, \wedge, \vee, \rightharpoonup, \top, \bot \rangle$$

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and a Boolean filter F of $\mathbf{H}$ *let*

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}.$$

Then we have:

1. $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \to, \bot, \top\rangle$ *is a Nelson lattice, when the operations are defined as follows:*
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \rightharpoonup t) \wedge (s \rightharpoonup y))$,
   - $(x, y) \to (s, t) = ((x \rightharpoonup s) \wedge (t \rightharpoonup y), x \wedge t)$,
   - $\top = (\top, \bot)$, $\bot = (\bot, \top)$.
2. $\neg(x, y) = (y, x)$,
3. *Given a Nelson lattice* $\mathbf{A}$*, there is a Heyting algebra* $\mathbf{H}_{\mathbf{A}}$*, unique up to isomorphisms, and a unique Boolean filter* $F_{\mathbf{A}}$ *of* $\mathbf{H}_A$ *such that* $\mathbf{A}$ *is isomorphic to* $\mathbf{R}(\mathbf{H}_{\mathbf{A}}, F_{\mathbf{A}})$.

Remark 3. Let $\mathbf{A}$ be a Nelson lattice. Let us consider $H = \{a^2 : a \in \mathbf{A}\}$ with the operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in \mathbf{A}$. Then,

$$\mathbf{H}^*_{\mathbf{A}} = \langle H, \vee^*, \wedge^*, \to^*, 0, 1\rangle$$

is a Heyting algebra ([6]).

Now, for our aim, we need to introduce some definitions of modal algebras.

Definition 4. A modal Heyting algebra M$\mathbf{H}$ is an algebra $\langle \mathbf{H}, \Box, \Diamond\rangle$ such that the reduct $\mathbf{H}$ is an Heyting algebra, $\Box$ and $\Diamond$ are two binary operators and, for all $x, y \in H$,

$$\Box x \wedge \Diamond(-x \wedge y) = \bot \tag{1}$$

Modal Heyting algebras obviously form a variety but it is not very well known. However, there is well known extension of this that is called *normal* modal Heyting algebra. It is obtaned by including the following equations:

3. $-\Diamond x = \Box - x$,
4. $\Box(x \rightharpoonup y) \rightharpoonup (\Box x \rightharpoonup \Box y) = \top$,
5. $\Box\top = \top$.

Note that (1) implies that $\Box x \wedge \Diamond - x = \bot$ and $\Box - x \wedge \Diamond x = \bot$, therefore, we can conclude $\Diamond - x \leq -\Box x$ and $\Box - x \leq -\Diamond x$. In addition, if (5) is assumed, we have $\Diamond\bot = \bot$.

Definition 5. A modal N3-lattice (for short MN3-lattice) is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge\rangle$ such that the reduct $\mathbf{A}$ is an N3-lattice and, for all $a, b \in A$,

1. $\blacklozenge a = \neg\blacksquare\neg a$,
2. $(\blacksquare a)^2 = (\blacksquare a^2)^2$ and $(\blacklozenge a)^2 = (\blacklozenge a^2)^2$,

3. $(\blacksquare a \wedge \blacklozenge(\neg a^2 \wedge b))^2 = \bot$.

In addition, **A** is said to be regular if it satifies the following:

4. $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Moreover, if **A** is a regular modal N3-lattice (for short RMN3-lattice) by using (1) and (4), we can conclude:

4′. $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$.

Finally, we say that a modal Nelson lattice is normal if it is regular and, in addition, satisfies:

5. $\blacksquare\top = \top$.

In this case, we can reproduce the following classical result on RMN3-lattices:

Lemma 6. *If* **A** *is a regular modal N3-lattice then it satisfies the next monotony properties:*

$$\text{if } a^2 \leq b \text{ then } (\blacksquare a)^2 \leq \blacksquare b, \quad \text{and} \quad \text{if } (\neg a)^2 \leq \neg b \text{ then } (\neg\blacksquare a)^2 \leq \neg\blacksquare b.$$

Now we are ready to formulate the first result of this work.

Theorem 7. *Let* **H** *and* F *be a modal Heyting algebra as defined in 4 and a Boolean filter satisfying:*

$$\text{if } x \wedge y = \bot \text{ and } x \vee y \in F \text{ then } \Box x \vee \Diamond y \in F.$$

Then, $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \to, \bot, \top, \blacksquare, \blacklozenge\rangle$ *is a Modal Nelson lattice, where the operators* $\blacksquare, \blacklozenge$ *are defined as follows:*

$$\blacksquare(x, y) = (\Box x, \Diamond y), \quad \text{and} \quad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

Now, we are going to extend the representation of Nelson lattices in terms of Heyting algebras from Theorem 2 to the modal context. First, we need to introduce the next result.

Lemma 8. *Let* **A** *be a MN3-lattice. Consider* $\mathbf{H}^*_{\mathbf{A}} = \langle H, \vee^*, \wedge^*, \to^*, \bot, \top, \Box^*, \Diamond^*\rangle$ *with* $H = \{a^2 : a \in A\}$ *and operators* $\vee^*, \wedge^*, \to^*$ *as in Remark 3 and modal operators as follows*

$$\Box^* a = (\blacksquare a)^2, \quad \text{and} \quad \Diamond^* a = (\blacklozenge a)^2$$

for every $a \in H$*. Then* $\mathbf{H}^*_{\mathbf{A}}$ *is a modal Heyting algebra. In addition, if we take* $F = \{(a \vee \neg a)^2 : a \in A\}$*, then* F *is a Boolean filter satisfying*

$$\text{if } a \vee^* b \in F \text{ and } a \wedge^* b = \bot \text{ then } \Box^* a \vee^* \Diamond^* b \in F$$

for every $a, b \in H$.

A direct consequence of previous Lemma is our main result:

Theorem 9. *Let* **A** *be a modal N3-lattice. Then* **A** *is isomorphic to* $\mathbf{R}(\mathbf{H}^*_{\mathbf{A}}, F)$ *as defined in Theorem 2 by taking* F *as in the previous lemma.*

Now, we would like to finish our presentation by considering two interesting subvariaties of Modal Nelson lattices. First, we consider the modal extension of the subvariety of Nelson lattices, introduced in [3] which is characterized by the following equation:

$$\neg a^2 \to a^2 = (\neg a \to a)^2 \tag{2}$$

We denote this modal subvariety by $\mathcal{MNN}$. Let us consider a modal Heyting algebra $\langle \mathbf{H}, \Box, \Diamond\rangle$ such that for all $x \in H$ the following conditions hold:

1. $--\Box x = -\Diamond - x$, 2. $-\Box - x = --\Diamond x$.

Then, we are able to prove the following result.

Theorem 10. *A modal Nelson lattice $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ satisfies Equation (2) if and only if there is a modal Heyting algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$ that satisfies conditions* 1. *and* 2. *such that $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ is isomorphic to $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$.*

Now, we are going to extend the notion of ϕ-regular algebra to the modal context. The subvariety of Nelson lattices called ϕ-regular Nelson lattices were studied in [1] which is characterized by:

$$(\neg a^2)^2 \vee (\neg(\neg a^2)^2)^2 = \top \tag{3}$$

We are going to say that $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ is a modal ϕ-regular Nelson algebra if the non-modal reduct of $\mathbf{A}$ is ϕ-regular algebra where for any $a \in A$ the unary term $\phi(a) = (\neg(\neg a)^2)^2 \wedge (\neg(\neg(a \vee \neg a)^2)^2 \vee a)$ satisfies:

$$\blacksquare\phi(a) = \phi(\blacksquare a),$$

Definition 11. A modal Heyting algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$ is said to be *crisp-witnessed* if it satisfies the equations

$$--\Box x = \Box --x \quad \text{and} \quad --\Diamond x = \Diamond --x \tag{4}$$

for every $x \in H$.

Thus, we are able to prove the following result.

Theorem 12. *A modal Nelson lattice $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ is a modal ϕ-regular Nelson lattice if and only if the associated Heyting algebra $\langle \mathbf{H}_\mathbf{A}, \Box, \Diamond \rangle$ is crisp-witnessed and satisfies the Stone identity $-x \vee --x = \top$.*

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