The Algebra of Indicative Conditionals[*]

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$\wedge_{\mathsf{OL}}$	0	1/2	1
0	0	0	0
1/2	0	1/2	1
1	0	1	1

$\vee_{\mathsf{OL}}$	0	1/2	1
0	0	0	1
1/2	0	1/2	1
1	1	1	1

	$\neg$
0	1
1/2	1/2
1	0

$\wedge_{\mathsf{K}}$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\vee_{\mathsf{K}}$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\to_{\mathsf{OL}}$	0	1/2	1
0	1/2	1/2	1/2
1/2	0	1/2	1
1	0	1/2	1

$\to_{\mathsf{DF}}$	0	1/2	1
0	1/2	1/2	1/2
1/2	1/2	1/2	1/2
1	0	1/2	1

$\to_{\mathsf{F}}$	0	1/2	1
0	1/2	1/2	1/2
1/2	0	1/2	1/2
1	0	1/2	1

Figure 1: Tables of the three-valued connectives.

Indicative conditionals are the simplest sentences of the *if-then* type that occur in natural language, concerning what could be true – in opposition to counterfactuals, which concern eventualities that are no longer possible. In Boolean propositional logic, an indicative conditional "if φ then ψ" is traditionally formalized as the material implication $\varphi \to \psi$, or equivalently the disjunction $\neg\,\varphi \vee \psi$. This approach has several limitations that have been remarked early on in the history of modern logic: in particular, a number of authors argued that conditionals having a false antecedent – which are true in Boolean logic independently of the consequent – should instead be regarded as lacking a (classical) truth value. Such a proposal can be traced back at least to Reichenbach (1935), De Finetti (1936), and Quine (1950). "Uttering a conditional amounts to making a *conditional assertion*: the speaker is committed to the truth of the consequent when the antecedent is true, but committed to neither truth nor falsity of the consequent when the antecedent is false" [1, p. 188]; see also [2] and the references cited therein.

Among various possible ways to formalize the above intuition, a very simple one consists in expanding the classical truth values $(\mathbf{0}, \mathbf{1})$ with a third "gap" value (here denoted by $\mathbf{1/2}$) assigned to conditional sentences with a false antecedent; and then extending the truth tables of the propositional connectives in accordance with the above interpretation. In particular, with regard to the implication, one would certainly require $\mathbf{0} \to x = \mathbf{1/2}$, whereas in other cases (e.g. $\mathbf{1/2} \to x$) intuitions may differ (see Figure 1). As for the designated elements to be preserved in derivations, it is natural to include (besides $\mathbf{1}$) also $\mathbf{1/2}$, at least if one wants to retain basic classical tautologies such as the law of identity $(\varphi \to \varphi)$.[1]

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[1]A peculiar consequence of this setup is that there will be valid formulas whose negation is also valid: for instance the formula $\neg\,\varphi \to (\varphi \to \varphi)$, which turns out to be equivalent (within the systems considered here) to $\mathbf{1/2}$ viewed as a propositional constant. This makes the logics under consideration not only paraconsistent but actually *contradictory* in the sense of Wansing [13].

The above constraints determine a range of three-valued propositional *logics of indicative conditionals* which turn out to be, in general, not subclassical (i.e. weaker than) but rather incomparable with classical two-valued logic. In particular, they may be *connexive* in that they validate the (classically contingent) formulas known as *Aristotle's* thesis $\neg(\varphi \to \neg\varphi)$ and *Boethius'* theses: $(\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$ and $(\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$.

Logics of indicative conditionals are discussed at length in the papers [1, 2, 3], which are the main bibliographical source and the starting point for the present research. Here we consider these propositional systems from the standpoint of algebraic logic: in particular, we determine which among them are algebraizable in the sense of Blok and Pigozzi [4], and study the corresponding algebra-based semantics. Besides the ones discussed in [1, 2], we shall also define a few systems obtained by varying the above-mentioned basic parameters (in particular, the designated elements) that do not appear to have been considered in the existing literature; our interest in the latter logics is essentially formal, but future research may prove them to be also relevant to the issues discussed above.

As is well known, a standard way of introducing a propositional logic is to fix an algebra $\mathbf{A}$ together with a subset $D \subseteq A$ of *designated elements* to be preserved in derivations. Such a pair $\langle \mathbf{A}, D\rangle$ is known as a *(logical) matrix*[2], and we may unambiguously denote by $\mathrm{Log}\langle \mathbf{A}, D\rangle$ the propositional consequence relation determined by the matrix $\langle \mathbf{A}, D\rangle$. For the logics of interest here, the universe of the algebra is always going to be the three-element set $A_3 = \{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}$, with variations only in the algebraic operations considered, and possibly the set of designated values. The basic systems are the following (in all cases, we fix $D = \{\mathbf{1/2}, \mathbf{1}\}$):

1. $\mathrm{Log}\langle \mathbf{DF_3}, D\rangle$, where $\mathbf{DF_3} = \langle A_3; \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{DF}}, \neg\rangle$, which is the logic proposed by De Finetti [5]. We show that, up to definitional equivalence, this system coincides with Priest's *logic of paradox* LP [6] expanded with the propositional constant $\mathbf{1/2}$.

2. $\mathrm{Log}\langle \mathbf{OL_3}, D\rangle$, where $\mathbf{OL_3} = \langle A_3; \wedge_{\mathsf{OL}}, \vee_{\mathsf{OL}}, \to_{\mathsf{OL}}, \neg\rangle$. This is the structural weakening of Cooper's *logic of ordinary discourse* [7], dubbed sOL in the recent papers [8, 9].

3. $\mathrm{Log}\langle \mathbf{CN_3}, D\rangle$, where $\mathbf{CN_3} = \langle A_3; \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{OL}}, \neg\rangle$. A system introduced by Cantwell [10] as the *logic of conditional negation* (CN) and independently considered by a number of other authors[3]. We prove that CN may be viewed as a term-definable subsystem of sOL.

4. $\mathrm{Log}\langle \mathbf{F_3}, D\rangle$, where $\mathbf{F_3} = \langle A_3; \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{F}}, \neg\rangle$, a logic introduced by Farrell [11]. We show that this system is definitionally equivalent to CN (hence, also to a definable subsystem of sOL).

Besides the above systems, we consider a few related ones that, as far as we are aware, have not yet appeared in the literature. These are obtained by:

5. Varying the set D of designated elements on A_3: for instance, logics that result from taking $D = \{\mathbf{1/2}\}$, which is a natural choice at least from a formal standpoint.

6. Considering a set of matrices based on the same algebra. In this way we study *degree-preserving logics* associated to the above-mentioned algebras (see e.g. [12]).

In each case we determine whether the system is algebraizable, thereby settling some issues on the algebraization of logics of indicative conditionals that were raised but left unsolved in [2]. Algebraizable logics are well-behaved in many ways, and in particular one may easily obtain a presentation of the algebraic semantics from an axiomatization of the logic, and vice-versa. In these cases we produce such axiomatizations, and also introduce *twist representations* (akin

[2] See, e.g., [14] for further background on the theory of logical matrices.

[3] As pointed out in [17], this logic – or equivalent systems, with slight variations in the choice of primitive connectives – seems to have been introduced independently in a number of papers from the 1980s to the 2000s (see, e.g., [15, 16]).

to that in [9]) that provide further insight into the algebraic semantics; in all the other cases we nevertheless employ algebraic logic techniques to try and obtain some understanding of the models of the logic under consideration.

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