Degree of Kripke Incompleteness in Tense Logics

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1 Introduction

Kripke-completeness of modal logics has been extensively studied since 1960s. Thomason [14] established the existence of Kripke-incomplete tense logics, that is, tense logics which are not complete with respect to any class of Kripke frames. Later, Fine [9] and van Benthem [15] found examples of Kripke-incomplete modal logics. Fine [9] raised a question concerning the degree of Kripke-incompleteness of logics in the lattice $\mathsf{NExt}(\mathsf{K})$ of all normal modal logics. In general, for each lattice $\mathcal{L}$ of logics and $L \in \mathcal{L}$, the *degree of Kripke-incompleteness* $\mathsf{deg}_{\mathcal{L}}(L)$ *of* L *in* $\mathcal{L}$ is defined as:

$$\mathsf{deg}_{\mathcal{L}}(L) = |\{L' \in \mathcal{L} : \mathsf{Fr}(L') = \mathsf{Fr}(L)\}|.$$ [1]

In other words, the degree of Kripke-incompleteness of L in $\mathcal{L}$ is the cardinality of logics in $\mathcal{L}$ which share the same class of Kripke-frames with L. A logic L is *strictly Kripke-complete in* $\mathcal{L}$ if $\mathsf{deg}_{\mathcal{L}}(L) = 1$. A celebrated result on Kripke-incompleteness is the dichotomy theorem for degree of Kripke-incompleteness in $\mathsf{NExt}(\mathsf{K})$ by Blok [3]: every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$. This theorem was proved in [3] algebraically by showing that union splittings in $\mathsf{NExt}(\mathsf{K})$ are exactly the consistent strictly Kripke-complete logics and all other consistent logics have the degree $2^{\aleph_0}$. A proof based on Kripke semantics was given later in [4]. This characterization of the degree of Kripke-incompleteness indicates locations of Kripke-complete logics in the lattice $\mathsf{NExt}(\mathsf{K})$.

Further results have been obtained on generalizations of degree of Kripke-incompleteness. The degree of modal incompleteness with respect to neighborhood semantics was investigated in [8, 11, 5]. Dziobiak [8] proved the dichotomy theorem for degree of incompleteness in the lattice $\mathsf{NExt}(\mathsf{D} \oplus (\Box^n p \to \Box^{n+1} p))$ w.r.t neighborhood semantics for all $n \in \omega$. Litak [11] studied modal incompleteness w.r.t Boolean algebras with operators (BAOs) and showed the existence of a continuum of neighborhood-incomplete modal logics extending Grz. For more on modal incompleteness from an algebraic view, we refer the readers to [12]. Degree of finite model property (FMP) was introduced in [1], where the following anti-dichotomy theorem for the degree of FMP for extensions of the intuitionistic propositional logic IPC was proved: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{Ext}(\mathsf{IPC})$ such that the degree of FMP of L in $\mathsf{Ext}(\mathsf{IPC})$ is κ. It was also shown in [1] that the anti-dichotomy theorem of the degree of FMP holds for $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. Degrees of FMP in bi-intuitionistic logics were studied in [7].

It is a longstanding open problem whether Blok's dichotomy theorem holds for extensions of transitive modal logics such as $\mathsf{K4}$ and $\mathsf{S4}$, or for extensions of the intuitionistic logic IPC. Since the Blok's proof relies on non-transitive frames heavily, we need new technique to solve these problems.

Tense logics are bi-modal logics that include a future-looking necessity modality $\Box$ and a past-looking possibility modality $\blacksquare$, of which the lattices are substantially different from those of modal logics (see [10, 14, 13]). However, as far as we know, no characterization of the degree of Kripke-incompleteness in lattices of tense logics is known. In this work, we study Kripke-incompleteness in tense logics. We start with the lattice $\mathsf{NExt}(\mathsf{K4}_t)$ of transitive tense logics. Inspired by the proof for Blok's dichotomy theorem in [4], we prove the dichotomy theorem for transitive tense logics, that is, every tense logic $L \in \mathsf{NExt}(\mathsf{K4}_t)$ is of degree of Kripke-incompleteness 1 or $2^{\aleph_0}$. By a similar argument, we also show that dichotomy theorem of the degree of Kripke-incompleteness holds for $\mathsf{NExt}(\mathsf{K}_t)$.

[1] To simplify notation, we always write deg_{L_0} for $\mathsf{deg}_{\mathsf{NExt}(L_0)}$.

2 Main Results

Let $L^* = \mathsf{K4}_t \oplus (\Diamond\top \vee \blacklozenge\top)$ and $L^\circ = \mathsf{K}_t \oplus (\Diamond\top \vee \blacklozenge\top)$. Our main result is the following theorem:

Theorem 1. *Let $L \in \mathsf{NExt}(\mathsf{K}_t)$. Then the following holds:*

(1) If $L \in \{\mathsf{K}_t, L^\circ\}$, then $\mathsf{deg}_{\mathsf{K}_t}(L) = 1$. Otherwise $\mathsf{deg}_{\mathsf{K}_t}(L) = 2^{\aleph_0}$.

(2) Suppose $L \in \mathsf{NExt}(\mathsf{K4}_t)$. If $L \in \{\mathsf{K4}_t, L^*\}$, then $\mathsf{deg}_{\mathsf{K4}_t}(L) = 1$. Otherwise $\mathsf{deg}_{\mathsf{K4}_t}(L) = 2^{\aleph_0}$.

Dichotomy theorems for tense logics and transitive tense logics follow from Theorem 1. An interesting corollary is that even the inconsistent tense logic $\mathscr{L}_t$ is of degree of Kripke-incompleteness $2^{\aleph_0}$, which means that there are continuum many logics in $\mathsf{NExt}(\mathsf{K4}_t)$ with no Kripke frame.

3 Proof Idea

In what follows, we report on the proof idea of Theorem 1(2) and the main technique used.

Definition 2. *A* Kripke frame *is a pair $\mathfrak{F} = (X, R)$ where $X \neq \varnothing$ and $R \subseteq X \times X$. The* inverse *of R is defined as $\breve{R} = \{\langle v, w\rangle : wRv\}$. For every $w \in X$, let $R[w] = \{u \in X : wRu\}$ and $\breve{R}[w] = \{u \in X : uRw\}$. For every $U \subseteq W$, we define $R[U] = \bigcup_{x\in U} R[x]$ and $\breve{R}[U] = \bigcup_{x\in U} \breve{R}[x]$.*

For $k \geq 0$, we define $R^k_\sharp[w]$ by $R^0_\sharp[w] = \{w\}$ and $R^{k+1}_\sharp[w] = R^k_\sharp[w] \cup R[R^k_\sharp[w]] \cup \breve{R}[R^k_\sharp[w]]$. Let $R^\omega_\sharp[w] = \bigcup_{k\geq 0} R^k_\sharp[w]$. For all binary relation R, we write R^+ for its transitive closure.

Intuitively, $R^n_\sharp[w]$ is the set of all points which can be reached from w by an $(R \cup \breve{R})$-path of length no more than n. Models, truth and validity of tense formulas are defined as usual. For each $n \in \omega$ and $\varphi \in \mathscr{L}_t$, we define the formula $\Delta^n\varphi$ by: $\Delta^0\varphi = \varphi$ and $\Delta^{k+1}\varphi = \Delta^k\varphi \vee \Diamond\Delta^k\varphi \vee \blacklozenge\Delta^k\varphi$. Then the readers can verify that $\mathfrak{M}, w \models \Delta^n\varphi$ iff $\mathfrak{M}, u \models \varphi$ for some $u \in R^n_\sharp[w]$.

Lemma 3. *Let $L \in \mathsf{NExt}(\mathsf{K4}_t)$. Then $L \in \{\mathsf{K4}_t, L^*\}$ implies $\mathsf{deg}_{\mathsf{K4}_t}(L) = 1$.*

Proof. By Kripke-completeness of $\mathsf{K4}_t$, $\mathsf{deg}_{\mathsf{K4}_t}(\mathsf{K4}_t) = 1$. To show $\mathsf{deg}_{\mathsf{K4}_t}(L^*) = 1$, suppose there exists $L' \in \mathsf{NExt}(\mathsf{K4}_t)$ such that $\mathsf{Fr}(L') = \mathsf{Fr}(L^*)$ and $L' \neq L^*$. Since L^* is Kripke-complete, $L' \subsetneq L^*$ and so $\Diamond\top \vee \blacklozenge\top \notin L'$. Thus $(\{0\}, \varnothing) \in \mathsf{Fr}(L')$, which contradicts to $\mathsf{Fr}(L') = \mathsf{Fr}(L^*)$. ☐

To prove the second half of Theorem 1(2), we need some auxiliary frame-constructions which can bring us frames containing long enough zigzags. *In what follows, by frames we mean rooted transitive frames.* Let us recall the book-construction of frames from [10, Section 3]. Consider frames $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ such that $X \cap Y = \{u\}$. Then $\mathfrak{H} = ((X \cup Y), (R \cup S)^+)$ is a frame such that $\mathfrak{H}{\upharpoonright}X \cong \mathfrak{F}$ and $\mathfrak{H}{\upharpoonright}Y \cong \mathfrak{G}$. Since we can always re-label points in domains of frames, by similar idea, for all frames $\mathfrak{F} = (X, R)$, $\mathfrak{G} = (Y, S)$ and points $w \in X$ and $u \in Y$, we can construct the *combination* $\langle \mathfrak{F}w{+}u\mathfrak{G}\rangle$ *of* $(\mathfrak{F}, w)$ *and* $(\mathfrak{G}, u)$ by 'gluing' $\mathfrak{F}$ and $\mathfrak{G}$ at w and u.

Let $\mathfrak{F} = (X, R)$ be a frame. For any $n \in \mathbb{Z}^+$ and fixed points $w, u \in X$, the n-pages book $\mathfrak{F}^n_{w,u}$ of $\mathfrak{F}$ is constructed by combining n copies of $\mathfrak{F}$ at w and u alternatively. An example of the book construction is given in Figure 1. It is not hard to verify that $\mathfrak{F}$ is a t-morphic image of $\mathfrak{F}^n_{w,u}$ for each $n \in \mathbb{Z}^+$. As a corollary, for all $x \in X$ and $n \in \mathbb{Z}^+$, $\mathsf{Th}(\mathfrak{F}^n_{w,u}, x) \subseteq \mathsf{Th}(\mathfrak{F}, x)$.[2] Moreover, if $\langle w, u\rangle \in R \setminus \breve{R}$, then the book-construction can provide us frames with long enough zigzags. Formally, the following lemma holds:

Lemma 4. *Let $\mathfrak{F} = (X, R)$, $w, u \in X$, Rwu and $u \notin R[w]$. Let $n \in \omega$ and $\mathfrak{G} = (Y, S) = \mathfrak{F}^{4n+2}_{w,u}$. Then $S^n_\sharp[x] \neq Y$ holds for all $x \in X$.*

Now we start to prove the second half of Theorem 1(2). Let $L \in \mathsf{NExt}(\mathsf{K4}_t)$ be an arbitrarily fixed logic such that $L \notin \{\mathsf{K4}_t, L^*\}$. Then $L \nsubseteq L^*$. Take any $\varphi_L \in L \setminus L^*$. Then we can show that φ_L is refuted by some finite non-symmetric frame. By Lemma 4, we can prove the following lemma:

[2] For all frames $\mathfrak{F} = (X, R)$ and $x \in X$, we define $\mathsf{Th}(\mathfrak{F}, x) = \{\varphi \in \mathscr{L}_t : \mathfrak{F}, x \models \varphi\}$.

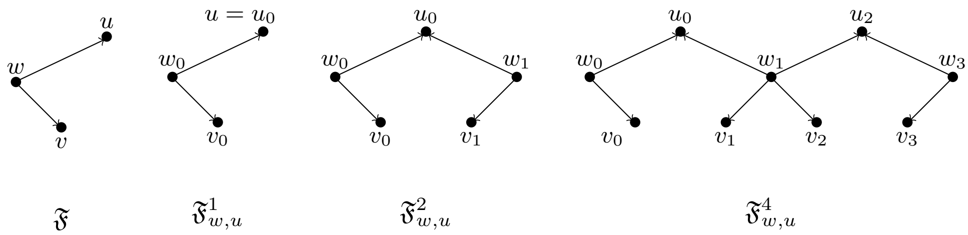

Figure 1: Examples for the book construction

Lemma 5. *There is a finite frame* $\mathfrak{F}_L$ *and* $w_L, u_L \in X$ *such that* $\mathfrak{F}_L, w_L \not\models \varphi_L$ *and* $u_L \notin R_\sharp^{\mathsf{md}(\varphi)}[w_L]$.

Let $\mathbb{Z}^\flat = \omega \setminus \{0,1\}$. For each $I \in \mathcal{P}(\mathbb{Z}^\flat)$, we define the general frame $\mathbb{F}_I = (X_I, R_I, P_I)$ as follows:

- $X_I = X_L \uplus (\omega \cup \{i^* : i \in I\})$.
- $R_I = (R_L \cup \{\langle n, m\rangle \in \omega \times \omega : n < m\} \cup \{\langle i^*, i\rangle : i \in I\} \cup \{\langle 0, u_L\rangle\})^+$.
- P_I is the tense algebra generated by $\mathcal{P}(X_L)$.

Example 6. *Let* $\mathbb{P}$ *be the set of all prime numbers. Then* $\mathfrak{F}_\mathbb{P}$ *is depicted by Figure 2.*

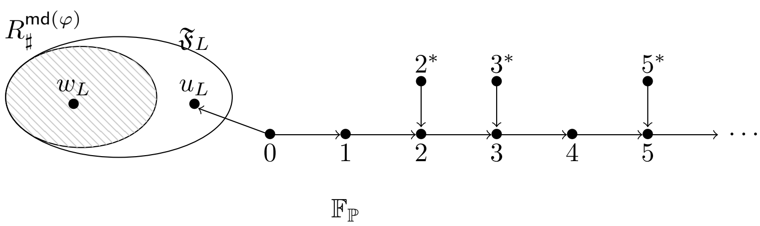

Figure 2: The frame $\mathbb{F}_\mathbb{P}$

Let $I \in \mathcal{P}(\mathbb{Z}^\flat)$ be arbitrarily fixed and take the minimal $k \in \omega$ such that $|\mathfrak{F}_L| < k$ and $X_I = {R_I}_\sharp^k[v]$ for all $v \in X_I$. For each $n \in \omega$ and $m \in \mathbb{Z}^\flat$, we define the formulas γ_n and γ_m^* as follows:

- $\gamma_0 = \blacksquare\bot \wedge \Diamond\blacksquare^2\bot \wedge \Diamond^k\blacksquare^{k+1}\bot$ and $\gamma_{l+1} = \blacklozenge\gamma_l \wedge \blacksquare^2\neg\gamma_l$.
- $\gamma_m^* = \Diamond\gamma_m \wedge \Box\neg\gamma_{m-1} \wedge \blacksquare\bot$.

Then we can verify that for all $n \in \omega$ and $m \in I$, the constant formulas γ_n and γ_m^* are true at exactly points n and m^*, respectively. Let $L_I = \mathsf{Log}(\mathsf{Fr}(L) \cup \{\mathbb{F}_I\})$. Clearly, $L_I \subseteq \mathsf{Log}(\mathsf{Fr}(L))$ and so $\mathsf{Fr}(L) = \mathsf{Fr}(\mathsf{Log}(\mathsf{Fr}(L))) \subseteq \mathsf{Fr}(L_I)$. Note that for all distinct $I, J \in \mathcal{P}(\mathbb{Z}^\flat)$, $L_I \neq L_J$. Indeed, take any $I \not\subseteq J$ and $i \in I \setminus J$, we can show that $\neg\varphi_L \to \Delta^k\gamma_i^* \in L_I \setminus L_J$. Moreover, we have

Lemma 7. $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ *for all* $I \in \mathbb{Z}^\flat$.

Proof. (Sketch.) Suppose $\mathsf{Fr}(L) \neq \mathsf{Fr}(L_I)$. Then $\mathsf{Fr}(L) \subsetneq \mathsf{Fr}(L_I)$ and there is a frame $\mathfrak{G} = (Y, S) \in \mathsf{Fr}(L_I)$ and $y \in Y$ such that $\mathfrak{G}, y \not\models \psi$ for some $\psi \in L$. Thus $\mathbb{F}_I \models \exists^k(\gamma_0 \wedge \Diamond\gamma_1)$ and so $\neg\psi \to \exists^k(\gamma_0 \wedge \Diamond\gamma_1) \in L_I$. Since $\psi \in L$, $\mathbb{F}_I, 0 \models \Box(\Box p \to p) \to \Box p$ and $\mathbb{F}_I, 0 \models \Box(\gamma_i \to \Diamond\gamma_{i+1})$ for all $i \in \omega$, we have

$$\exists^k\neg\psi \wedge \gamma_0 \to (\Box(\Box p \to p) \to \Box p) \in L_I \text{ and } \{\exists^k\neg\psi \wedge \gamma_0 \to \Box(\gamma_i \to \Diamond\gamma_{i+1}) : i \in \omega\} \subseteq L_I.$$

Since $\mathfrak{G} \not\models \psi$ and $\mathfrak{G} \models \neg(\gamma_i \leftrightarrow \gamma_j)$ for any different $i, j \in \omega$, there exists an infinite strict S-chain $\langle u_i : i \in \omega \rangle \subseteq U$ such that $y = u_0$ and $\mathfrak{G}, u_i \models \gamma_i$ for all $i \in \omega$. Take any propositional variable $p \in \mathsf{Prop}$ which does not occur in ψ. Then we see that $\mathfrak{G}, u_0 \not\models \Box(\Box p \to p) \to \Box p$. Hence $\mathfrak{G} \not\models \exists^k \neg\psi \wedge \gamma_0 \to (\Box(\Box p \to p) \to \Box p)$, which contradicts $\mathfrak{G} \models L_I$. $\Box$

Since $I \in \mathcal{P}(\mathbb{Z}^\flat)$ is arbitrarily fixed and $|\mathcal{P}(\mathbb{Z}^\flat)| = 2^{\aleph_0}$, we conclude that $\mathsf{deg}_{\mathsf{K4}_t}(L) = 2^{\aleph_0}$. Note that $L \notin \{\mathsf{K4}_t, L^*\}$ is also chosen arbitrarily, the proof of Theorem 1(2) is concluded.

To show Theorem 1(1), take any $L \notin \{\mathsf{K}_t, L^\circ\}$. Then $L \nsubseteq L^\circ$ and there exists $\varphi_L \in L \setminus L^\circ$. By the non-transitive book-construction in [10] or the unrevealing construction introduced in [2], Lemma 5 holds. For each $I \in \mathcal{P}(\mathbb{Z}^\flat)$, we define the general frame $\mathbb{F}'_I = (X_I, R'_I, P_I)$, where $R_I = R_L \cup \{\langle n, m\rangle : n < m\} \cup \{\langle i^*, j\rangle : i \in I \text{ and } i \leq j\} \cup \{\langle 0, u_L\rangle\}$. By similar arguments, $L'_I = \mathsf{Log}(\mathsf{Fr}(L) \cup \{\mathbb{F}'_I\})$ share the same frames with L and $|\{L'_I : I \subseteq \mathbb{Z}^\flat\}| = 2^{\aleph_0}$.

In fact, Theorem 1 is also a generalization of Blok's characterization of the degree of Kripke-incompleteness of modal logics. It follows from [10, Theorem 22] that $\{\mathsf{K}_t, L^\circ\}$ and $\{\mathsf{K4}_t, L^*\}$ are the sets of union splittings in $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$, respectively. Thus we have

Theorem 8. *Let $L_0 \in \{\mathsf{K}_t, \mathsf{K4}_t\}$ and $L \in \mathsf{NExt}(L_0)$ be consistent. If L is a union splitting in $\mathsf{NExt}(L_0)$, then $\mathsf{deg}_{L_0}(L) = 1$. Otherwise $\mathsf{deg}_{L_0}(L) = 2^{\aleph_0}$.*

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