Uniform validity of atomic Kreisel-Putnam rule in monotonic proof-theoretic semantics

In recent years, many completeness or incompleteness results have been proved for variants of *monotonic proof-theoretic semantics* (mPTS), see [1, 2, 6, 7, 8, 12, 11, 13, 14]. A model of mPTS is a set $\mathfrak{B}$ of *atomic rules R of level $n \geq 0$* of the form

$$\dfrac{\begin{matrix}[\mathfrak{R}_1] \\ A_1\end{matrix} \qquad \dots \qquad \begin{matrix}[\mathfrak{R}_n] \\ A_n\end{matrix}}{A}\, R$$

where A_i, A are atoms from the underlying language, and $\mathfrak{R}_i$ is an atomic rule of level $n-2$ discharged by R ($i \leq n$). That A is *a consequence of* Γ *on* $\mathfrak{B}$, written $\Gamma \models_{\mathfrak{B}} A$, is defined thus—limiting ourselves to propositional logic:

Definition 1. $\Gamma \models_{\mathfrak{B}} A \Longleftrightarrow$

- $\Gamma = \emptyset \Longrightarrow$
  - A *is atomic* $\Longrightarrow \vdash_{\mathfrak{B}} A$
  - $A = B \wedge C \Longrightarrow\ \models_{\mathfrak{B}} B$ *and* $\models_{\mathfrak{B}} C$
  - $A = B \vee C \Longrightarrow\ \models_{\mathfrak{B}} B$ *or* $\models_{\mathfrak{B}} C$
  - $A = B \to C \Longrightarrow B \models_{\mathfrak{B}} C$
- $\Gamma \neq \emptyset \Longrightarrow \forall \mathfrak{C} \supseteq \mathfrak{B}\ (\models_{\mathfrak{C}} \Gamma \Longrightarrow\ \models_{\mathfrak{C}} A)$.

Definition 2. $\Gamma \models A \Longleftrightarrow \forall \mathfrak{B}\ (\Gamma \models_{\mathfrak{B}} A)$.

—for approaches where one also takes care of constraints on the specific kinds of (sets of) sets of atomic rules of given levels, see [8, 12, 13, 14].

These variants of mPTS have been qualified as *sentential*—see [12]—since they start with a primitive notion of consequence on $\mathfrak{B}$, and so differ from Prawitz's original approach [9, 10], where *consequence of A from Γ on $\mathfrak{B}$* is *defined* as existence of an argument from Γ to A valid on $\mathfrak{B}$—logical consequence is likewise defined as existence of a logically valid argument from Γ to A. Valid arguments on $\mathfrak{B}$ are here prior, and defined as follows (some preliminary definitions are required).

Definition 3. *An* argument structure *is a tree $\mathscr{D}$ with nodes labelled by formulas, and whose edges are arbitrary inferences that may discharge top-formulas or atomic rules. The undischarged top-formulas are the* assumptions of *$\mathscr{D}$, while the root is the* conclusion of *$\mathscr{D}$.*

Definition 4. *$\mathscr{D}$ is* closed *if all its assumptions are discharged, and it is* open *otherwise.*

Definition 5. *$\mathscr{D}$ is canonical if it ends by applying an introduction, and it is* non-canonical *otherwise.*

Definition 6. *An* inference rule *R is a set of argument structures.*

I write $\mathscr{D}[\mathscr{D}^{**}/\mathscr{D}^{*}]$ the result of replacing by $\mathscr{D}^{**}$ the sub-structure $\mathscr{D}^{*}$ of $\mathscr{D}$.

Definition 7. *Let σ be a function from formulas A to (closed) argument structures with conclusion A. The* (closed) σ-instance *$\mathscr{D}^{\sigma}$ of $\mathscr{D}$ is $\mathscr{D}[\sigma(A_1), ..., \sigma(A_n)/A_1, ..., A_n]$, where $A_1, ..., A_n$ are the undischarged assumptions of $\mathscr{D}$.*

Definition 8. *A* reduction for *R is a function ϕ from argument structures to argument structures, defined on some $\mathbb{D} \subseteq R$ and such that, $\forall \mathscr{D} \in \mathbb{D}$,*

- *$\mathscr{D}$ is from Γ to $A \Longrightarrow \phi(\mathscr{D})$ is from $\Gamma^* \subseteq \Gamma$ to A*
- *$\forall \sigma$, ϕ is defined on $\mathscr{D}^\sigma$, and $\phi(\mathscr{D}^\sigma) = \phi(\mathscr{D})^\sigma$.*

Definition 9. *Let $\mathfrak{J}$ be a set of reductions. $\mathscr{D}$* reduces to *$\mathscr{D}^*$* relative to *$\mathfrak{J}$, written $\mathscr{D} \leq_{\mathfrak{J}} \mathscr{D}^*$, if there is a sequence $\mathscr{D}_1, ..., \mathscr{D}_n$ such that $\mathscr{D} = \mathscr{D}_1$, $\mathscr{D}^* = \mathscr{D}_n$ and, $\forall i \leq n$, $\mathscr{D}_{i+1} = \mathscr{D}_i[\phi(\mathscr{D}^{**})/\mathscr{D}^{**}]$ with $\phi \in \mathfrak{J}$.*

Definition 10. *$\langle \mathscr{D}, \mathfrak{J} \rangle$ is* valid on *$\mathfrak{B} \Longleftrightarrow$*

- *$\mathscr{D}$ is closed $\Longrightarrow$*
  - *the conclusion A of $\mathscr{D}$ is atomic $\Longrightarrow \mathscr{D} \leq_{\mathfrak{J}} \mathscr{D}^*$ with $\mathscr{D}^*$ closed derivation of A in $\mathfrak{B}$*
  - *$\mathscr{D}$ is canonical $\Longrightarrow$ the immediate sub-structures of $\mathscr{D}$ are valid on $\mathfrak{B}$ when paired with $\mathfrak{J}$*
  - *$\mathscr{D}$ is non-canonical $\Longrightarrow \mathscr{D} \leq_{\mathfrak{J}} \mathscr{D}^*$ with $\mathscr{D}^*$ canonical valid on $\mathfrak{B}$ when paired with $\mathfrak{J}$*
- *$\mathscr{D}$ is open with assumptions $A_1, ..., A_n \Longrightarrow \forall \mathfrak{C} \supseteq \mathfrak{B}, \mathfrak{H} \supseteq \mathfrak{J}, \sigma$, $(\langle \sigma(A_i), \mathfrak{H} \rangle$ valid on $\mathfrak{C} \Longrightarrow \langle \mathscr{D}^\sigma, \mathfrak{H} \rangle$ valid on $\mathfrak{C}$).*

Definition 11. *$\langle \mathscr{D}, \mathfrak{J} \rangle$ is* logically valid *$\Longleftrightarrow$ it is valid on every $\mathfrak{B}$.*

One might wonder whether (logical) consequence in sentential mPTS implies (logical) consequence in Prawitz's original approach, and vice versa. In [5] it is shown that, when the notion of set of reductions is liberal enough, then the two approaches do coincide, whence intuitionistic logic is incomplete relative to a Prawitzian-oriented framework with liberal sets of reductions.

In a stricter reading, sets of reductions are *base-independent* partial constructive functions for uniform rewriting of argument structures. It is not easy to define base-independence precisely, but the rough idea is that sets of reductions are constructive (or possibly finite) sets of operations whose output values are described by relying only on structural properties of the input values and, more in particular, with no reference to specific atomic bases and structures which input values reduce to on those bases—this seems to be how Prawitz thinks of reductions in [9, 10], but see also [4, 12]. Following [12], I shall call *uniform monotonic proof-theoretic semantics* (umPTS) this approach.

It is not clear how to attain a direct translation from mPTS to umPTS, mainly due to the fact that it is not clear how to prove in umPTS the mPTS clause for the open consequence case, i.e., with $\Gamma \neq \emptyset$,

$$\Gamma \models_{\mathfrak{B}} A \Longleftrightarrow \forall \mathfrak{C} \supseteq \mathfrak{B} \, (\models_{\mathfrak{C}} \Gamma \Longrightarrow \models_{\mathfrak{C}} A).$$

While the $\Longrightarrow$ direction of this clause trivially holds in umPTS too, the $\Longleftarrow$ might fail. The fact that the existence of closed $\langle \mathscr{D}, \mathfrak{J} \rangle$-s for the elements of Γ valid on $\mathfrak{C}$ implies the existence of a closed $\langle \mathscr{D}^*, \mathfrak{H} \rangle$ for A also valid on $\mathfrak{C}$, may not imply in any straightforward way the existence of an open $\langle \mathscr{D}^{**}, \mathfrak{K} \rangle$ from Γ to A valid on $\mathfrak{B}$, when $\mathfrak{K}$ is constrained to be uniform.

In my talk, however, I show that this issue can be overcome, and that incompleteness of intuitionistic logic with respect to umPTS can be actually proved without any need of translating results from mPTS to umPTS. This is done along the lines of some incompleteness proofs given in [1, 2, 7], which apply to a framework where the notion of validity is defined in terms of (intuitionistic) constructions.

As said, sets of reductions in umPTS are partial constructive functions of a special kind, so proofs referred to a notion of validity defined in terms of (intuitionistic) constructions may not be automatically transferable to umPTS. A rough attempt at turning the kind of constructions used in the incompleteness proofs from [1, 2, 7] into sets of reductions in umPTS, seems in fact to suggest that the sets of reductions thereby obtained would not be base-independent.

On the other hand, Pezlar [3] has recently proved that a generalised Kreisel-Putnam rule—where the antecedent of the premise is a Harrop formula—can be constructively justified through a selector with schematic elimination and equality type-theoretic rules. The formulation of Pezlar's selector in a Natural Deduction formalism seems to hint at the fact that something similar might be done for umPTS and, hence, that an incompleteness proof along the lines of [1, 2, 7] can be found also for Prawitz's original framework—monotonically understood. I will prove that this is the case for the intuitionistically underivable Kreisel-Putnam rule restricted to atoms, i.e.,

$$\dfrac{p \to q \lor r}{(p \to q) \lor (p \to r)}\ \mathtt{KP}a$$

by exhibiting a set of reductions for this rule which is clearly base-independent—however the concept of base-independence is defined—so the following is provable.

Theorem 1. *There is a base-independent set of reductions* $\mathfrak{J}$ *such that the open argument structure* $\mathtt{KP}a$ *is valid on every* $\mathfrak{B}$.

The required set of reductions is made up of the five functions $\phi_1, \phi_{2,i}, \phi_{3,i}$ defined as follows.

$$\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathscr{D}\\ B \lor C\end{matrix}}{A \to B \lor C}\to_I, 1}{(A \to B) \lor (A \to C)} \quad \overset{\phi_1}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{matrix}\overline{A}\\ \mathscr{D}\\ B \lor C\end{matrix}}{A \to B \lor C}\to_I}{(A \to B) \lor (A \to C)}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\overline{A}\\ \mathscr{D}\\ B_i\end{matrix}}{B_1 \lor B_2}\lor_{I,i}}{A \to B_1 \lor B_2}\to_I}{(A \to B_1) \lor (A \to B_2)} \quad \overset{\phi_{2,i}}{\Longrightarrow} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathscr{D}\\ B_i\end{matrix}}{B_1 \lor B_2}\lor_{I,i}}{A \to B_1 \lor B_2}\to_I, 1}{(A \to B_1) \lor (A \to B_2)}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathscr{D}\\ B_i\end{matrix}}{B_1 \lor B_2}\lor_{I,i}}{A \to B_1 \lor B_2}\to_I, 1}{(A \to B_1) \lor (A \to B_2)} \quad \overset{\phi_{3,i}}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathscr{D}\\ B_i\end{matrix}}{A \to B_i}\to_I, 1}{(A \to B_1) \lor (A \to B_2)}\lor_{I,i}$$

where $i = 1, 2$. From this we immediately draw the following conclusion.

Corollary 1. *Intuitionistic logic is incomplete over* umPTS.

When Prawitz's completeness conjecture from [10] is formulated in a monotonic approach, umPTS is the kind of proof-theoretic semantics which the conjecture should refer to, whence Corollary 1 implies a refutation of the conjecture for this "orthodox" Prawitzian framework.

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