On split extensions of hoops

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Abstract

Internal actions have been introduced in [3] by F. Borceux, G. Janelidze, and G. M. Kelly as a means to generalize the connection between actions and split extensions from groups and Lie algebras to arbitrary semi-abelian categories. However, in certain settings such as Orzech categories of interest [20] internal actions are often expressed in terms of external actions, i.e., via a set of maps which satisfy a certain set of identities. In this talk, we are gonna study external actions and split extensions in the category Hoops of hoops, with a focus on those split extensions which strongly splits. In particular, we say that a split extension

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

strongly splits, or has a *strong section*, if the morphism s is a *strong section* of p, i.e. if the equation

$$a \to s(b) = sp(a) \to s(b)$$

holds for every $a \in A$ and $b \in B$.

When a split extension

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

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in Hoops strongly splits, then the semidirect product $X \ltimes_\xi B$ of X and B with respect to the corresponding internal action ξ is given by the set

$$\{(x,b) \in X \times B \mid \; s(b) \to (s(b) \cdot x) = x\}$$

together with the operations

$$(x,b) \to (y,b') = (s(b' \to b) \to (x \to y),\; b \to b'),$$

$$(x,b) \cdot (y,b') = (s(b \cdot b') \to (s(b \cdot b') \cdot x \cdot y),\; b \cdot b')$$

and

$$1_{X \rtimes_\xi B} = (1,1,1).$$

Split extensions with strong section in the category Hoops can be described in terms of *strong external actions* [19], i.e., a pair of maps

$$f \colon B \times X \to X \colon (b,x) \mapsto f_b(x),$$

$$g \colon B \times X \to X \colon (b,x) \mapsto g_b(x)$$

such that

E1. $f_b(1) = g_b(1) = 1$;

E2. $f_1 = g_1 = \mathrm{id}_X$;

E3. $f_{b_1 \cdot b_2}(x \cdot g_{b_1}(x \to y)) = f_{b_1 \cdot b_2}(x \cdot (x \to y))$;

E4.

$$g_{(b_3 \to (b_1 \cdot b_2))}(f_{b_1 \cdot b_2}(x \cdot y) \to z) = \\ = g_{(b_2 \to b_3) \to b_1}(x \to g_{b_3 \to b_2}(y \to z));$$

for any $b, b_1, b_2, b_3 \in B$ and $x, y, z \in X$.

In particular, there is a bijection τ_B between the set $\mathrm{EAct}_{\mathrm{ss}}(B,X)$ of strong external actions of B on X and the set $\mathrm{SplExt}_{\mathrm{ss}}(B,X)$ of isomorphism classes of split extensions of B by X that strongly splits. Indeed, we can define τ_B as the map that sends every split extension in Hoops that strongly splits

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B,$$

to the pair of maps $f, g \colon B \times X \to X$ defined by

$$f_b(x) = s(b) \to (s(b) \cdot x) \;\text{ and }\; g_b(x) = s(b) \to x.$$

It is possibile to show that (f,g) defines a strong external action of B on X. Moreover, the map μ_B which sends a strong external action $f, g \colon B \times X \to X$ to the split extension of B by X

$$X \xrightarrow{\iota_1} Y \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} B$$

where

$$Y = \{(x,b) \in X \times B \mid \; f_b(x) = x\}$$

and

$$(x,b) \to (y,b') = (g_{b' \to b}(x \to y),\; b \to b'),$$
$$(x,b) \cdot (y,b') = (f_{b \cdot b'}(x \cdot y),\; b \cdot b')$$

is the inverse of the map τ_B.

As a consequence, there is a natural isomorphism

$$\tau\colon \operatorname{SplExt}_{ss}(-,X) \cong \operatorname{EAct}_{ss}(-,X),$$

where $\operatorname{SplExt}_{ss}(-,X)\colon \mathsf{Hoops}^{op} \to \mathsf{Set}$ is the functor which assigns to any hoop B, the set $\operatorname{SplExt}_{ss}(B,X)$ and

$$\operatorname{EAct}_{ss}(-,X)\colon \mathsf{Hoops}^{op} \to \mathsf{Set}$$

is the functor which maps every hoop B to $\operatorname{EAct}_{ss}(B,X)$.

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