Modal FL_{ew}-algebra satisfiability through first-order translation *

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Abstract

Modal logics offer a valid treatment for temporal and spatial data, which are critical in modeling many real-world scenarios and, therefore, are becoming more popular by the day in artificial intelligence applications, specifically when dealing with symbolic machine learning. Some notable examples are [10, 11], introducing modal logics for treating interval temporal relations and topological (i.e., spatial) relations, respectively. However, practitioners handling temporal and spatial data typically encounter challenges, as sensing and discretizing signals that often introduce inaccuracies in the data. Fuzzy logics are renowned as a common approach to deal with uncertainty and unclear boundaries in the data. Furthermore, Melvin Fitting proposed in [6] a many-valued approach leveraging Heyting algebras to tackle many-expert scenarios, another compelling application in artificial intelligence. In this talk, we want to present a framework that is general enough to treat modal many-valued logics, including Fitting's proposal, and can be endowed with reasoning tools suitable for real-world applications.

FL_{ew}-algebras (*Full Lambek calculus with exchange and weakening*, see, e.g., [8]) proved to be a valid candidate, as it generalizes most common algebraic structures of many-valued logics, such as *Gödel* algebras (G, for short) [1], *MV*-algebras [3] (MV) on which *Łukasiewicz* logic is based [12], *product* algebras (Π) [9], and *Heyting* algebras (H). FL_{ew}-algebras are *bounded integral commutative residuated lattices*; i.e., an FL_{ew}-algebra $\boldsymbol{A}$ is a *lattice* ordered by a partial ordering relation $\leq$, with a top (1) and a bottom (0) element. When the order is linear, we use the term *FL_{ew}-chain*. The difference between FL_{ew}-algebras and common bounded lattices is the presence of an internal operation, usually denoted by $\cdot$, and assumed to be commutative, associative and having 1 as a neutral element, usually referred to as *t-norm*, that is, such that $(\boldsymbol{A}, \cdot, 1)$ is a *monoid*; hence, we will often refer to the multiplication as the *monoidal operation*. Intuitively, the multiplication in an FL_{ew}-algebra generalizes the interpretation of the logical conjunction. Moreover, an FL_{ew}-algebra is assumed to have the *residuation property*, that is, it is assumed that for fixed elements $a, b \in \boldsymbol{A}$, there exists a unique maximal element x such that $a \cdot x \leq b$; this element is denoted by $a \to b$, and the implication operator $\to$ generalizes the logical implication. All commonly used algebras

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in the field of many-valued logics are particular cases of some FL_{ew}-algebra $(\boldsymbol{A}, \cdot, \rightarrow, 1, 0)$; each specific case differs from the others in how the monoidal operation is defined.

While *modal* many-valued logics [4, 6] have already been applied in different contexts, they are just starting to be studied in depth. Automated theorem proving for modal FL_{ew}-algebra formulas encompassing a many-valued generalization of Halpern and Shoham's interval temporal logic [10] has been tackled in [2] using a tableaux system inspired by the one proposed by Melvin Fitting in [7] and already extended to Heyting Algebras in [5]. When tackling formulas satisfiability and validity in FL_{ew}-algebras (and, more generally, many-valued logics defined over a lattice representing a partial order), the problem can be relaxed to finding, given a formula φ and a value α in the algebra, if a model exists such that (resp., for all possible models) the formula has at least value α. This problem is referred to α-satisfiability (resp. α-validity).

In this work, we propose a different approach leveraging well-known sat and smt solvers, such as *z*3, with the hope of gaining better performance while maintaining some sort of interpretability. In order to do so, one has to translate the α-satisfiability problem to a two-sorted first-order problem, with a first sort $\mathcal{A}$ representing the values in the FL_{ew}-algebra and a second sort $\mathcal{W}$ representing the worlds, such that given a formula φ interpreted on an FL_{ew}-algebra A, φ is α-satisfiable if and only if it exists $\mathcal{M}, w \in \mathcal{W}$ so that $\mathcal{V}_{\mathcal{M}}(w, a) \succeq \alpha$.

We provide an accessible and open-source algorithmic tool for (*i*) defining finite FL_{ew}-algebras, (*ii*) writing formulas in a specified FL_{ew}-algebra, and (*iii*) asking α-satisfiability for a given value α in the algebra of the formula through a first-order translation and making use of a sat or a smt solver, such as *z*3. This tool is offered as part of a long-term open-source framework for learning and reasoning, namely Sole.jl[1]. In particular, the tool can be found in the ManyValuedLogics submodule of the SoleLogics.jl[2] package, which provides the core data structures and functions for an easy manipulation of propositional, modal and many-valued logics. For the benefit the reader, the tool is also available in a standalone repository[3], using many-valued Halpern and Shoham's interval temporal logic as an example.

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[1] https://github.com/aclai-lab/Sole.jl
[2] https://github.com/aclai-lab/SoleLogics.jl
[3] https://github.com/aclai-lab/LATD2025b

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