Bochvar algebras, Płonka sums, and twist products

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Abstract

The proper quasivariety $\mathcal{BCA}$ of Bochvar algebras, which serves as the equivalent algebraic semantics of Bochvar's external logic, was introduced by Finn and Grigolia in [6] and extensively studied in [4]. We show that the algebraic category of Bochvar algebras is equivalent to a category whose objects are pairs consisting of a Boolean algebra and a meet-subsemilattice (with unit) of the same. We also show that one of the functors that induce the equivalence can be equivalently defined either by means of a Płonka sum construction, or by means of a twist product construction.

1 Extended abstract

In 1938, the Russian mathematician Dmitri Anatolyevich Bochvar published the influential paper "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus"[1], in which he introduced a 3-valued logic aimed at resolving set-theoretic and semantic paradoxes. His proposal diverged significantly from other related approaches in at least two key aspects. First and foremost, the third, non-classical value $\frac{1}{2}$ was *infectious* in any sentential compound involving the standard, or *internal*, propositional connectives $\neg, \wedge, \vee$. This means that a formula would be assigned the value $\frac{1}{2}$ iff at least one variable within it was assigned $\frac{1}{2}$. This third value was interpreted as "paradoxical". Second, the language of Bochvar's logic included *external* unary connectives J_0, J_1, J_2 (written in Finn and Grigolia's notation) that, unlike the internal connectives, could output only Boolean values.

Although the merits of Bochvar's logic as a solution to the paradoxes remain highly debatable, its influence on successive developments in 3-valued logic has been significant. The internal fragment of Bochvar's logic was characterised by Urquhart [9] through the imposition of a variable inclusion strainer on the consequence relation of classical propositional logic. Building on this result, a general framework for *right variable inclusion logics* has been proposed (see [3] for a detailed account). In this context, the celebrated algebraic construction of *Płonka sums* is extended from algebras to logical matrices. Specifically, each logic L is paired with a "right variable inclusion companion" L^r whose matrix models are decomposed as Płonka sums of models of L. Notably, Bochvar's internal logic serves as the right variable inclusion companion of classical logic.

Studies on Bochvar's external logic, by contrast, are comparatively scarce. Finn and Grigolia [6] provided an algebraic semantics for it with respect to the quasivariety of *Bochvar algebras*. However, their work does not employ the standard toolbox or terminology of abstract algebraic logic. Adopting a more mainstream approach, the papers [2, 4] extend Finn and Grigolia's completeness theorem to a full-fledged algebraisability result, and offer a representation of Bochvar algebras that refines the Płonka sum representations of their involutive bisemilattice reducts. We present Bochvar algebras in a simplified signature where the definable operation symbols J_0, J_1 are omitted.

Definition 1. *A Bochvar algebra is an algebra* $\mathbf{A} = \langle A, \wedge, \vee, \neg, J_2, 0, 1\rangle$ *of type* $\langle 2, 2, 1, 0, 0\rangle$ *that satisfies the following identities:*

1. $\varphi \vee \varphi \approx \varphi$;
2. $\varphi \vee \psi \approx \psi \vee \varphi$;
3. $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$;
4. $\varphi \wedge (\psi \vee \delta) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$;
5. $\neg(\neg\varphi) \approx \varphi$;
6. $\neg 1 \approx 0$;
7. $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$;
8. $0 \vee \varphi \approx \varphi$;
9. $J_2\neg J_2\varphi \approx \neg J_2\varphi$;
10. $J_2\varphi \approx \neg(J_2\neg\varphi \vee \neg(J_2\varphi \vee J_2\neg\varphi))$;
11. $J_2\varphi \vee \neg J_2\varphi \approx 1$;
12. $J_2(\varphi \vee \psi) \approx (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
13. $J_2\neg\varphi \approx J_2\neg\psi \ \& \ J_2\varphi \approx J_2\psi \ \Rightarrow \ \varphi \approx \psi$.

We show that the algebraic category of Bochvar algebras is equivalent to a category whose objects are pairs consisting of a Boolean algebra and a meet-subsemilattice (with unit) of the same. This equivalence instantiates the general theory of adjunctions between quasivarieties proposed by Moraschini [7].

Definition 2. *A* Bochvar system *is a pair* $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ *such that* $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1 \rangle$ *is a Boolean algebra and* $\mathbf{I} = \langle I, \wedge, 1 \rangle$ *is a meet-subsemilattice with unit of* $\mathbf{B}$.

Let $\mathfrak{B}$ denote the algebraic category of Bochvar algebras. We now define a category $\mathfrak{S}$ whose objects are Bochvar systems. If $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{I}_1 \rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{I}_2 \rangle$ are objects in $\mathfrak{S}$, a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$ is a homomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in I_2$ for every $i \in I_1$. Observe that any such g is also a homomorphism from $\mathbf{I}_1$ to $\mathbf{I}_2$.

Theorem 1. *The categories* $\mathfrak{B}$ *and* $\mathfrak{S}$ *are equivalent.*

Proof. (Sketch.) Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ be a Bochvar system. We define

$$\mathbb{A}_{\mathbb{B}} = \left\langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}^{\partial}, \{p_{ij} : i \leq_{\mathbf{I}^{\partial}} j\} \right\rangle$$

such that:

- for all $i \in I$, $\mathbf{A}_i := \mathbf{B}/[i)$;
- $\mathbf{I}^{\partial}$ is the lower-bounded join-semilattice dual to $\mathbf{I}$;
- for all $i, j \in I$ such that $i \leq_{\mathbf{I}^{\partial}} j$, $p_{ij}(a/[i)) := (a/[i))/[j)$.

$\mathbb{A}_{\mathbb{B}}$ is a semilattice direct system of Boolean algebras, whence the Płonka sum $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ over it is an involutive bisemilattice [3, Ch. 2]. By the results in [4], this is the underlying involutive bisemilattice of a unique Bochvar algebra, noted $\mathbf{A}_{\mathbb{B}}$.

For the other direction, let

$$\mathbf{A} = \langle A, \wedge, \vee, \neg, J_2, 0, 1\rangle$$

be a Bochvar algebra, whose involutive bisemilattice reduct decomposes as $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$. We define $\mathbb{B}_{\mathbf{A}} := \langle \mathbf{A}_{i_0}, \mathbf{K}\rangle$, where $K = \{J_2^{\mathbf{A}}(1^{A_i}) : i \in I\}$, and for $J_2^{\mathbf{A}}(1^{A_i}), J_2^{\mathbf{A}}(1^{A_j}) \in K$, $J_2^{\mathbf{A}}(1^{A_i}) \leq_{\mathbf{K}} J_2^{\mathbf{A}}(1^{A_j})$ iff $j \leq_{\mathbf{I}} i$. We have that $\mathbb{B}_{\mathbf{A}}$ is a Bochvar system.

We now define the map Γ as follows:

- If $\mathbf{A}$ is an object in $\mathfrak{B}$, let $\Gamma(\mathbf{A}) := \mathbb{B}_{\mathbf{A}}$.
- If $f : \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in $\mathfrak{B}$, let $\Gamma(f)$ be the restriction of f to $\mathbf{A}_{1_{i_0}}$.

Similarly, we define the map Ξ as follows:

- If $\mathbb{B}$ is an object in $\mathfrak{S}$, let $\Xi(\mathbb{B}) := \mathbf{A}_{\mathbb{B}}$.
- If $g : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathfrak{S}$, let $\Xi(g)$ be defined as follows: $\Xi(g)(a/[i]) := g(a)/[g(i)]$.

Γ and Ξ are functors that induce an equivalence between $\mathfrak{B}$ and $\mathfrak{S}$. □

Interestingly, the functor Ξ can be equivalently defined by resorting not to a Płonka-type construction, but rather to the definition of a *twist product algebra*.

Definition 3. *Let* $\mathbb{B} = \langle \mathbf{B}, \mathbf{I}\rangle$ *be a Bochvar system. The* twist product algebra *over* $\mathbb{B}$ *is the algebra*

$$Tw(\mathbb{B}) = \langle T, \wedge^{Tw(\mathbb{B})}, \vee^{Tw(\mathbb{B})}, \neg^{Tw(\mathbb{B})}, J_2^{Tw(\mathbb{B})}, 0^{Tw(\mathbb{B})}, 1^{Tw(\mathbb{B})}\rangle$$

of type $\langle 2, 2, 1, 0, 0\rangle$*, such that (omitting superscripts when denoting the operations in* $\mathbf{B}$*):*

- $T := \{\langle a, b\rangle : a, b \in B, a \wedge b = 0, a \vee b \in I\}$*;*
- $\langle a, b\rangle \wedge^{Tw(\mathbb{B})} \langle c, d\rangle := \langle a \wedge c, (b \wedge d) \vee (b \wedge c) \vee (a \wedge d)\rangle$*;*
- $\langle a, b\rangle \vee^{Tw(\mathbb{B})} \langle c, d\rangle := \langle (a \wedge c) \vee (a \wedge d) \vee (b \wedge c), b \wedge d\rangle$*;*
- $\neg^{Tw(\mathbb{B})}\langle a, b\rangle := \langle b, a\rangle$*;*
- $J_2^{Tw(\mathbb{B})}\langle a, b\rangle := \langle a, \neg a\rangle$*;*
- $0^{Tw(\mathbb{B})} := \langle 0, 1\rangle$*;*
- $1^{Tw(\mathbb{B})} := \langle 1, 0\rangle$*.*

Theorem 2. $Tw(\mathbb{B})$ *is a Bochvar algebra that is isomorphic to* $\mathbf{A}_{\mathbb{B}}$*.*

This observation might point both to a possible extension of the theory of twist products beyond the lattice-ordered case, and to a further exploration of the relationships between the constructions of Płonka sums and twist products. Moreover, it might relate to recent work on twist constructions and residuated lattices with conuclei [5, 8].

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