Computational complexity of satisfiability problems in Łukasiewicz logic

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In dealing with optimization problems, a crucial role is played by the maximal satisfiability problem of the Boolean logic, henceforth denoted by MaxSAT_B, as well as its weighted version, denoted by $\mathsf{WMaxSAT}_B$, and the partial (weighted) satisfiability problem $\mathsf{P(W)MaxSAT}_B$. These problems are quite relevant since they can have theoretical as well as practical applications. Indeed, $\mathsf{WMaxSAT}_B$ and MaxSAT_B are among the first examples of $\mathsf{FP}^{\mathsf{NP}}$ and $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$ complete problems[1] (see [4]). Furthermore, one of the most commonly used strategies to prove the hardness of a given problem with respect to the aforementioned complexity classes is to find a metric reduction of it starting from $\mathsf{WMaxSAT}_B$ or MaxSAT_B. At the same time, many real-world problems can be encoded using the $\mathsf{PMaxSAT}_B$ framework (see, for example, [1] for an application to data analysis).

The first attempts to study the maximum satisfiability problem within the context of Łukasiewicz logic, denoted as MaxSAT, can be found in [5] and [3]. The main motivation for this generalization, as outlined in the introduction of [5], is that Łukasiewicz logic offers a richer framework, allowing certain problems to be naturally expressed in this language– for instance, problems involving continuous variables. In [3], the authors proved that MaxSAT is $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$-complete.

In many real-life applications, it is often beneficial to prioritize the satisfiability of certain formulas over others. This can be achieved by assigning to each formula φ a weight a_φ, denoting its relative importance. Furthermore, Łukasiewicz logic allows us to deal with intermediate truth values between absolute falsity and absolute truth. Putting these two remarks together, the first results we present in this talk is a *(weighted) version* of the maximum satisfiability problem in Łukasiewicz logic. Specifically, we say that a formula φ is r-satisfiable, with $r \in (0, 1]$, if there exists a valuation v such that $v(\varphi) \geq r$. In the context of Łukasiewicz logic, it was proved in [6] that r-satisfiability is NP-complete. Furthermore, in the concluding remarks of [3], the authors defined the maximum r-satisfiability problem, $\mathsf{MaxSAT}_\mathsf{r}$, similarly to the MaxSAT problem, with the distinction that it focuses on the maximum number of formulas r-satisfied by a valuation. Hence, we consider the following problem.

Definition 1. Let $r \in (0, 1]$ be a rational number, and let F be a (non-empty) multiset of Łukasiewicz formulas. Let $0 \neq a_\varphi \in \mathbb{N}$ be the weight associated to the formula $\varphi \in F$. The $\mathsf{WMaxSAT}_\mathsf{r}$ problem is the optimization problem that computes the maximum $0 \leq k \leq \sum_{\varphi \in F} a_\varphi$ such that there exists a valuation v such that $k = \sum_{\varphi \in S} a_\varphi$, where $S \subseteq F$ is the multiset of all formulas r-satisfied by v.

The problem WMaxSAT is obtained from Definition 1 by fixing $r = 1$. The problem $\mathsf{MaxSAT}_\mathsf{r}$ is obtained from Definition 1 when all weights are equal to 1.

The first results we show are contained in next theorem.

[1] We recall that a function f belongs to the class $\mathsf{FP}^{\mathsf{NP}}$ if it is computable by a polynomial-bounded Turing machine with oracle NP. Similarly, f belongs to $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$ if $f \in \mathsf{FP}^{\mathsf{NP}}$ and f is computable using at most $O(\log(n))$ queries.

Theorem 2. *The following hold.*

1. $\mathsf{WMaxSAT}$ *is* $\mathsf{FP}^{\mathsf{NP}}$-*complete.*

2. $\mathsf{MaxSAT_r}$ *is* $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$-*complete.*

3. $\mathsf{WMaxSAT_r}$ *is* $\mathsf{FP}^{\mathsf{NP}}$-*complete.*

Specifically, Theorem 2(1) is proved by reducing the (weighted) satisfiability in Łukasiewicz logic to a MIP problem, similarly to what is done in [2]; Theorem 2(3) is proved using a metric reduction to WMaxSAT. The proof of Theorem 2(2) is inspired by the results of [3], where the authors define the problem but leave open the task of finding its computational complexity.

To conclude this talk, we introduce the *partial (weighted) r-satisfiability problem*, denoted by $\mathsf{P(W)MaxSAT_r}$, in the context of Łukasiewicz logic. Formally, we consider the following problem.

Definition 3. Let $r \in (0,1]$ be a rational number, and let H and S be two multiset of Łukasiewicz formulas, with $S \neq \emptyset$. Let $0 \neq a_\varphi \in \mathbb{N}$ be the weight associated to the formula $\varphi \in S$. The $\mathsf{PWMaxSAT_r}$ problem is the optimization problem that computes the maximum $0 \leq k \leq \sum_{\varphi \in S} a_\varphi$ such that there exists a valuation v such that $v[H] = 1$ and $k = \sum_{\varphi \in T} a_\varphi$, where $T \subseteq S$ the multiset of formulas r-satisfied by v. If H is not satisfiable, by definition the solution of $\mathsf{PWMaxSAT_r}$ is $-\infty$.

The problem PWMaxSAT is obtained from Definition 3 by fixing $r = 1$, and the problem $\mathsf{PMaxSAT_r}$ is obtained by Definition 3 when $a_\varphi = 1$ for all $\varphi \in S$. The following result can be proved by generalizing the arguments used to prove Theorem 2.

Theorem 4. *The following hold.*

1. $\mathsf{PWMaxSAT}$ *is* $\mathsf{FP}^{\mathsf{NP}}$-*complete.*

2. $\mathsf{PMaxSAT_r}$ *is* $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$-*complete.*

3. $\mathsf{PWMaxSAT_r}$ *is* $\mathsf{FP}^{\mathsf{NP}}$-*complete.*

We remark that $\mathsf{PWMaxSAT_r}$, as it happens for its Boolean counterpart, has a lot of potential to real-world applications. Indeed, any continuous non-linear functions in n variables $f(x_1, \ldots, x_n) : [0,1]^n \to [0,1]$ can be approximated using rational piecewise linear functions. By the results of [7], such functions can be represented by a pair (H, φ), where $H \cup \{\varphi\}$ is a set of Łukasiewicz formulas in the propositional variables Var and $\{x_1, \ldots, x_n\} \subseteq \mathsf{Var}$. This representation has the property that, for any continuous and piecewise function f represented by (H, φ), if v is a $[0,1]$-valued evaluation that satisfies H, then $f(v(x_1), \ldots, v(x_n)) = v(\varphi)$. Hence, $\mathsf{PWMaxSAT_r}$ can be used as a potential framework for the resolution of many optimization problems.

Finally, if time allows it, we explore appropriate versions of the satisfiability problems outlined above within the context of fuzzy probabilistic logics.

References

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