The Beth companion: making implicit operations explicit. Part II

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While arbitrary implicit operations are defined in terms of existential positive formulas, most familiar ones can be defined simply by conjunctions of equations. For instance,

1. the implicit operation of "taking the inverse y of x" is defined in monoids by
$$(x \cdot y \approx 1) \,\&\, (y \cdot x \approx 1);$$
2. that of "taking the complement y of x" is defined in bounded distributive lattices by
$$(x \wedge y \approx 0) \,\&\, (x \vee y \approx 1);$$
3. that of "taking the meet y of x_1 and x_2" is defined in Hilbert algebras by
$$(y \to x_1 \approx 1) \,\&\, (y \to x_2 \approx 1) \,\&\, (x_1 \to (x_2 \to y) \approx 1).$$

The following result explains, at least in part, why this is the case. To this end, given a class of algebras K and a term $t(x_1, \dots, x_n)$, we say that a first-order formula $\varphi(x_1, \dots, x_n, y)$ *defines* t in K when
$$\mathsf{K} \vDash t(x_1, \dots, x_n) \approx y \leftrightarrow \varphi(x_1, \dots, x_n, y).$$

Theorem 1. *Let K be a quasivariety with the amalgamation property and M a pp expansion of K. Then for each term $t(x_1, \dots, x_n)$ of M there exists a conjunction of equations $\varphi_t(x_1, \dots, x_n, y)$ in the language of K which defines t in M.*

The possibility of defining implicit operations in terms of conjunctions of equations (as opposed to arbitrary pp formulas) acquires special importance when applied to Beth companions.

Definition 2. *Let K be a quasivariety. A class of algebras M is said to be a* Beth completion *of K when it is a Beth companion and for each term $t(x_1, \dots, x_n)$ of M there exists a conjunction of equations $\varphi_t(x_1, \dots, x_n, y)$ in the language of K which defines t in M.*

For instance, Abelian groups, Boolean algebras, and implicative semilattices are Beth completions of the quasivarieties of commutative cancellative monoids, bounded distributive lattices, and Hilbert algebras, respectively.

In view of the following result, the Beth completion of a quasivariety K is unique (up to term-equivalence), when it exists. In this case, a class is a Beth completion of K iff it is a Beth companion of K.

*Speaker.

Proposition 3. *Let K be a quasivariety with a Beth completion. Then every Beth companion of K is a Beth completion of K. Furthermore, all the Beth completions of K are term-equivalent.*

From Theorem 1 we obtain the following sufficient condition for a Beth companion to be a Beth completion.

Theorem 4. *Let K be a quasivariety with the amalgamation property. If K has a Beth companion, then it also has a Beth completion.*

Notably, Beth completions retain many interesting properties from their original quasivariety. The next observation captures a few of them.

Proposition 5. *The following conditions hold for a quasivariety K with a Beth completion M:*

1. *if K is a variety, then so is M;*
2. *every lattice equation valid in the lattices of K-congruences of the members of K is also valid in the lattices of M-congruences of the members of M.*

Our aim is to show that, not only do Beth completions retain some of the properties of their original quasivarieties, but that these properties are often significantly enhanced in Beth completions (see Theorem 6).

To this end, we recall that a variety is said to be *arithmetical* when it is both congruence distributive and congruence permutable. Furthermore, we recall that a member $\mathbf{A}$ of a quasivariety K is *relatively finitely subdirectly irreducible* (RFSI for short) when the identity congruence of $\mathbf{A}$ is meet-irreducible in the lattice $\mathsf{Con}_{\mathsf{K}}(\mathbf{A})$ of K-congruences of K. The class of RFSI members of K will be denoted by K_{RFSI}. When K is a variety, we drop the "relatively" and write simply K_{FSI}. Lastly, a variety is said to have the *congruence extension property* when for each $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$, every congruence of $\mathbf{A}$ can be extended to a congruence of $\mathbf{B}$.

Our main result takes the following form.

Theorem 6. *Let K be a relatively congruence distributive quasivariety such that K_{RFSI} is closed under nontrivial subalgebras. The following conditions hold for every Beth completion M of K:*

1. *M is a variety;*
2. *M is arithmetical;*
3. *M has the congruence extension property;*
4. *M_{FSI} is closed under nontrivial subalgebras.*

In other words, in the Beth completion we gain the congruence extension property, as well as the following improvements:

$$\text{quasivariety} \longmapsto \text{variety};$$
$$\text{relative congruence distributivity} \longmapsto \text{arithmeticity}.$$

At the same time, we preserve our assumptions on the class of RFSI algebras.

For instance, Theorem 6 provides a general explanation of why, moving from arbitrary bounded distributive lattices to Boolean algebras, we gain congruence permutability. For recall that the class of Boolean algebras is the Beth completion of the class of bounded distributive lattices which, in turn, is amenable to Theorem 6. In view of condition 2, Boolean algebras must be arithmetical and, therefore, congruence permutable.