Generating and counting finite FL_{ew}-chains *

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Abstract

Toward a more systematic analysis of the several variants of supporting algebras for various kinds of propositional and modal many-valued logics, *FL_{ew}-algebras* (*Full Lambek calculus with exchange and weakening*, see, e.g., [11]) were introduced to generalize the most common algebraic structures, such as *Gödel* algebras (*G*, for short) [3], *MV*-algebras [7] (*MV*) on which *Łukasiewicz* logic is based [16], *product* algebras (Π) [14], and *Heyting* algebras (*H*) that may provide an infinitely-valued interpretation of *intuitionistic* logic [9, 13, 15]. Each of these logics offers unique capabilities that have proven beneficial across various disciplines, including mathematics, computer science, and particularly artificial intelligence, where they enhance expressive power and decision-making processes; this is particularly true in the case of *modal* many-valued logics [8, 10], which have already been applied in different contexts but are just starting to be studied in depth.

The structure of FL_{ew}-algebras is of interest for both mathematicians and computer scientists; indeed, FL_{ew}-algebras are precisely *bounded integral commutative residuated lattices*. This means that an FL_{ew}-algebra $\boldsymbol{A}$ is *lattice ordered* by a partial ordering relation $\leq$, with a top (1) and a bottom (0) element. When the order is linear, we use the term *FL_{ew}-chain*. The additional structure that distinguishes FL_{ew}-algebras from common bounded lattices is given by another internal operation, usually denoted by $\cdot$, and assumed to be commutative, associative and having 1 as neutral element, sometimes referred to as *t-norm*, that is, such that $(\boldsymbol{A}, \cdot, 1)$ is a *monoid*; hence, we will often refer to the multiplication as the *monoidal operation*. Intuitively, the multiplication in an FL_{ew}-algebra generalizes the interpretation of the logical conjunction. Moreover, an FL_{ew}-algebra is assumed to have the *residuation property*, that is, it is assumed that for any elements $a, b \in \boldsymbol{A}$, there exists a unique maximal element x such that $a \cdot x \leq b$; this element is denoted by $a \to b$, and the implication operator $\to$ generalizes the logical implication. Most commonly used algebras in the field of fuzzy and many-valued logics are particular cases of some FL_{ew}-algebra $(\boldsymbol{A}, \cdot, \to, 1, 0)$; each specific case differs from the others in how the monoidal operation is defined.

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While fuzzy logics are generally based on *infinite* algebras (typically built on the interval $[0, 1]$ of real numbers), the *finite* case is very interesting in practical cases [2]; among other contexts, datasets in machine learning are finite by definition, naturally leading to finite descriptions of patterns.

The question of probing a variety of finite algebras in order to count its non-isomorphic elements is a very natural one. De Baets and Mesiar [4] count the number of different t-norms that can be built on a chain of length n. Bartušek and Navara [5] solve the same problem by proposing a tool that actually generates all such t-norms. Belohlavek and Vychodil [6] again answer the question of generating all different residuated lattices, although, according to their definition, they actually focus on FL_{ew}-algebras of size n. Finally, Galatos and Jipsen [12] publish the set of all different FL_{ew}-algebras of size up to 6. Notwithstanding, the actual algorithm used for generation is published only in [6], and no database of FL_{ew}-algebras is actually current available for further analysis. Furthermore, no explicit bound for the number of different FL_{ew}-algebras has been given, and the numerical results are limited to the published constants.

In this work, we approach, again, the problem of counting and generating all different FL_{ew}-chains of size n, and, in particular: (*i*) we use a novel approach to this problem based on a topological interpretation of residuation theory, which shares some similarities with Scott's work in domain theory [1, 17]; (*ii*) we provide an explicit bound for the number of different FL_{ew}-chains of size n; (*iii*) we provide an accessible and open-source algorithmic tool for generating and counting FL_{ew}-chains as part of a long-term open-source framework for learning and reasoning, namely Sole.jl[1]. In particular, the tool can be found in the ManyValuedLogics submodule of the SoleLogics.jl[2] package, which provides the core data structures and functions for an easy manipulation of propositional, modal and many-valued logics. To ease the reader, the tool is also available in a standalone repository[3].

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