Recent Advances in Fundamental Logic

Guillaume Massas

Scuola Normale Superiore di Pisa
guillaume.massas@sns.it

Holliday [3] introduced a non-classical logic called *fundamental logic*, which captures exactly those properties of the connectives $\wedge, \vee$ and $\neg$ that hold in virtue of their introduction and elimination rules in Fitch's natural deduction system for propositional logic. Fundamental logic is a sublogic of both (the $\{\to\}$-free fragment of) intuitionistic logic and orthologic. The former can be obtained from fundamental logic by adding the *Reiteration* rule to Holliday's Fitch system for fundamental logic, while the second can be obtained by adding the *Double Negation Elimination* rule.

From the algebraic perspective, fundamental logic is the logical counterpart to the variety of fundamental lattices:

Definition 0.1. A *fundamental lattice* is a tuple $(L, \leq, \wedge, \vee, \neg, 0, 1)$ such that $(L, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice and $\neg : L \to L$ is an antitone map satisfying the following properties:

- $\neg 1 = 0$;
- $a \wedge \neg a = 0$;
- $a \leq \neg\neg a$.

Since fundamental logic is weaker than both intuitionistic logic and orthologic, fundamental lattices generalize both pseudocomplemented distributive lattices and ortholattices.

In this talk based on joint projects with Wes Holliday and Juan P. Aguilera respectively, I will present some recent results which shed some new light on the relationship between fundamental logic, intuitionistic logic and orthologic.

First, I will discuss two translations of fundamental logic into modal orthologic and modal intuitionistic logic. The first translation is based on the celebrated Gödel-McKinsey-Tarski translation [4] of intuitionistic logic into $\mathsf{S4}$, the modal logic of reflexive and transitive Kripke frames. The restriction of this translation to the $\{\to\}$-free fragment of IPC is a map τ inductively defined as follows:

$$\tau(p) = \Box p;$$

$$\tau(\neg\phi) = \Box\neg\tau(\phi);$$

$$\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi);$$

$$\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi).$$

As it turns out, this translation also yields an embedding of fundamental logic into $\mathsf{OS4}$, the natural counterpart of $\mathsf{S4}$ in orthomodal logic.

Theorem 0.2 (Holliday and Massas 2025)**.** *The Gödel-McKinsey-Tarski translation τ is a full and faithful translation of fundamental logic into* $\mathsf{OS4}$*.*

A similar result can be obtained by "swapping" the roles of intuitionistic logic and orthologic. Goldblatt [2] defined the following translation σ from the language of orthologic into the language of modal logic:

$$\sigma(p) = \Box\Diamond p;$$

$$\sigma(\neg\phi) = \Box\neg\sigma(\phi);$$

$$\sigma(\phi \wedge \psi) = \sigma(\phi) \wedge \sigma(\psi);$$

$$\sigma(\phi \vee \psi) = \Box\Diamond(\sigma(\phi) \vee \sigma(\psi)).$$

Goldblatt shows that σ is a full and faithful translation of orthologic into the modal logic KTB of reflexive and symmetric Kripke frames. In order to generalize this result, we define the logic FSTB, a natural counterpart of KTB in the setting of Fischer-Servi intuitionistic modal logics [1].

Definition 0.3. The intuitionistic modal logic FSTB extends the Fischer-Servi logic FS with the following axioms:

$$\Box\phi \vdash \phi,\ \phi \vdash \Diamond\phi;$$

$$\Diamond\Box\phi \vdash \phi,\ \phi \vdash \Diamond\Box\phi.$$

Theorem 0.4 (Holliday and Massas 2025)**.** *The Goldblatt translation σ is a full and faithful translation of fundamental logic into* FSTB.

These results establish that fundamental logic is, arguably, both "intuitionistic logic from the viewpoint of orthologic", and "orthologic from the viewpoint of intuitionistic logic".

Lastly, I will discuss the relationship between fundamental logic and orthointuitionistic logic, i.e., the strongest logic contained in both the $\{\to\}$-free fragment of intuitionistic logic and orthologic. Although fundamental logic is strictly weaker than orthointuitionistic logic, the latter turns out to have a reasonably simple axiomatization.

Definition 0.5. Let OI be the smallest consequence relation extending fundamental logic and closed under the following axioms:

- $p \wedge (q \vee r) \wedge \neg\neg s \vdash \neg\neg(p \wedge s) \wedge (s \vee (p \wedge q) \vee (p \wedge r))$;
- $\neg(((p \wedge q) \vee (p \wedge r)) \wedge s) \wedge s \vdash (p \wedge (q \vee r)) \vee \neg(p \wedge (q \vee r))$

Theorem 0.6 (Aguilera and Massas 2025)**.** *The logic* OI *is the strongest extension of fundamental logic that is weaker than both orthologic and intuitionistic logic.*

References

[1] Gisèle Fischer Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36:141–149, 1977.

[2] Robert I Goldblatt. Semantic analysis of orthologic. *Journal of Philosophical logic*, pages 19–35, 1974.

[3] Wesley H. Holliday. A fundamental non-classical logic. *Logics*, 1:36–79, 2023.

[4] J. C. C. McKinsey and Alfred Tarski. Some theorems about the sentential calculi of lewis and heyting. *The Journal of Symbolic Logic*, 13(1):1–15, 1948.