Projective Unification and Structural Completeness in Extensions and Fragments of Bi-Intuitionistic Logic

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1 Bi-Intuitionistic Logic

Definition 1. Let $\mathcal{L} \coloneqq (\wedge, \vee, \to, \leftarrow, \neg, \sim, \bot)$ be a formal algebraic language of signature $(2, 2, 2, 2, 1, 1, 0)$, called *bi-intuitionistic language*. Given a sublanguage $\mathcal{L}_1 \subseteq \mathcal{L}$, we denote its set of formulas built up from a denumerable set of variables $\{p, q, \dots\}$ by $Fm_{\mathcal{L}_1}$. If $\varphi \in Fm_{\mathcal{L}_1}$, then $var(\varphi)$ denotes the set of variables occurring in φ. A *logic* in $\mathcal{L}_1$ is a finitary consequence relation $\vdash$ on $Fm_{\mathcal{L}_1}$ that is also substitution invariant. We write $\Gamma \vdash \varphi$ instead of $(\Gamma, \varphi) \in \vdash$. If $\emptyset \vdash \varphi$ then φ is called a *theorem* of $\vdash$, and we will use the shorthand notation $\vdash \varphi$.

Let $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}$ be sublanguages. If $\vdash_1$ is a logic in $\mathcal{L}_1$, then the $\mathcal{L}_0$-*fragment* of $\vdash_1$ is the restriction of $\vdash_1$ to $Fm_{\mathcal{L}_0}$. If $\vdash_0$ is a logic in $\mathcal{L}_0$, then $\vdash_1$ is an *extension* of $\vdash_0$ if $\vdash_0 \subseteq \vdash_1$. If moreover $\mathcal{L}_0 = \mathcal{L}_1$ and there exists $\Sigma \subseteq Fm_{\mathcal{L}_0}$ such that

$$\Gamma \vdash_1 \varphi \iff \Gamma \cup \Sigma \vdash_0 \varphi$$

for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_0}$, we call $\vdash_1$ an *axiomatic extension* of $\vdash_0$. In the case that $\Sigma = \{\psi\}$, we will sometimes write $\vdash_1 = \vdash_0 + \psi$.

Bi-intuitionistic logic bi-IPC is the conservative extension of intuitionistic logic IPC obtained by enlarging the intuitionistic language with the connectives $\leftarrow$ and $\sim$, called *co-implication* and *co-negation*, and demanding that they behave dually to $\to$ and $\neg$, respectively. In this way, bi-IPC achieves a symmetry, which IPC lacks, between the connectives $\wedge, \to, \neg, \bot$ and $\vee, \leftarrow, \sim, \top$. The Kripke semantics of bi-IPC [10] provides a transparent interpretation of co-implication: given a Kripke model $\mathfrak{M}$, a point x in $\mathfrak{M}$, and formulas φ, ψ, then

$$\mathfrak{M}, x \models \varphi \leftarrow \psi \iff \exists y \leq x \ (\mathfrak{M}, y \models \varphi \text{ and } \mathfrak{M}, y \not\models \psi).$$

Using this equivalence, the intended behavior of co-negation also becomes clear, because the formula $\sim p \leftrightarrow (\top \leftarrow p)$ is a theorem of bi-IPC.

Equipped with these new connectives, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. In fact, Gödel's interpretation of IPC into the modal logic S4 can be extended to an interpretation of bi-IPC into the temporal modal logic tense-S4 [12].

The greater symmetry of bi-IPC with respect to IPC is reflected by the fact that bi-IPC is algebraized in the sense of [3] by the variety bi-HA of *bi-Heyting algebras* [9], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, we can (and will) identify bi-IPC with the logic induced by the class of matrices $\{\langle \mathbf{A}, \{1\}\rangle \colon \mathbf{A} \in \textsf{bi-HA}\}$, and sometimes denote it by $\vdash_{\textsf{bi-IPC}}$. Notably [6], for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have

$$\Gamma \vdash_{\textsf{bi-IPC}} \varphi \iff \text{if } \mathfrak{M} \text{ is a Kripke model, then } \mathfrak{M} \models \Gamma \text{ implies } \mathfrak{M} \models \varphi.$$

2 Projective Unification

Let $\mathcal{L}_0$ be a sublanguage of the bi-intuitionistic language $\mathcal{L}$ and $\vdash$ a logic in $\mathcal{L}_0$. A formula $\varphi \in Fm_{\mathcal{L}_0}$ is said to be *unifiable* in $\vdash$ if we have $\vdash \sigma(\varphi)$, for some substitution σ. In this case, σ is called a $\vdash$*-unifier* of φ, or simply a *unifier* of φ, when the logic $\vdash$ is clear from the context. If moreover the language $\mathcal{L}_0$ contains the connectives $\wedge$ and $\to$, and $\varphi \vdash p \leftrightarrow \sigma(p)$ holds for every $p \in var(\varphi)$, then σ is called a *projective unifier* of φ.

If σ and τ are two unifiers of φ, we say that σ is *at least as general as* τ, denoted by $\sigma \leq \tau$, if there exists a substitution μ such that $\vdash \sigma(p) \leftrightarrow \mu \circ \tau(p)$, for every $p \in var(\varphi)$. A set E of unifiers of φ is said to be a *basis* if: for every unifier τ of φ, there exists $\sigma \in E$ such that $\sigma \leq \tau$; and for all $\sigma, \sigma' \in E$, if $\sigma \leq \sigma'$ then $\sigma = \sigma'$. In particular, if $E = \{\sigma\}$ is a one-element basis, then σ is called a *most general unifier* of φ. It is easy to see that a projective unifier of φ is always a most general unifier of φ. We call φ *unitary* if it admits a most general unifier and *projective* if it admits a projective unifier. Accordingly, the logic $\vdash$ is said to be *unitary* (resp. *projective*) if every unifiable formula is unitary (resp. projective).

In the first part of this talk, I will present an unpublished joint work with Damiano Fornasiere and Quentin Gougeon, where we characterized the projective *bi-intermediate logics* (i.e., consistent axiomatic extensions of bi-IPC): they are exactly those which have a theorem of the form[1] $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$, for some $n \in \omega$. Compare this to [13], where it is shown that the projective *intermediate logics* (i.e., consistent axiomatic extensions of IPC) are exactly those which extend the *Gödel-Dummett logic* $\textsf{GD} \coloneqq \textsf{IPC} + (p \to q) \vee (q \to p)$. And although being an extension of the *bi-intuitionistic Gödel-Dummett logic* $\textsf{bi-GD} \coloneqq \textsf{bi-IPC} + (p \to q) \vee (q \to p)$ is a sufficient condition for a bi-intermediate logic to be projective (because $\neg\sim p \to (\neg\sim)^2 p$ is a theorem of bi-GD, see [2]), it is not necessary. For example, since $(\neg\sim)^2 p \to (\neg\sim)^3 p$ is a theorem of $\textsf{bi-IPC} + \neg\big((q \leftarrow p) \wedge (p \leftarrow q)\big)$, our characterization ensures that this bi-intermediate logic is projective, but it is not an extension of bi-GD [2].

Semantically, bi-intermediate logics with a theorem of the form $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$ can be characterized by the property of having a natural bound for the zigzag depth of the Kripke frames which validate them, a notion that we proceed to explain. If u and v are points in a Kripke frame $\mathfrak{F}$, we say that v *can be reached from* u *after* n*-many zigzags* if there are $x_1, y_1, \ldots, x_n \in \mathfrak{F}$ such that $u \leq x_1 \geq y_1 \leq x_2 \geq y_2 \leq \cdots \leq x_n \geq v$. We then define, for every $U \subseteq \mathfrak{F}$, the set $(\downarrow\uparrow)^n U$ of points of $\mathfrak{F}$ that can be reached from a point in U after n-many zigzags. Notably, if $\mathfrak{M} = (\mathfrak{F}, V)$ is a Kripke model on $\mathfrak{F}$, then $V\big((\neg\sim)^n \varphi\big) = (\downarrow\uparrow)^n V(\varphi)$ for all $\varphi \in Fm_{\mathcal{L}}$. Using this equality, one can easily show that

$$\mathfrak{F} \models (\neg\sim)^n p \to (\neg\sim)^{n+1} p \iff (\downarrow\uparrow)^n U = (\downarrow\uparrow)^{n+1} U \text{ for every upset } U \text{ of } \mathfrak{F},$$

and when these conditions are satisfied, we say that $\mathfrak{F}$ has n*-bounded zigzag depth.*

That a bi-intermediate logic $\vdash$ with a theorem of the form $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$ must be projective already follows from the literature: in [11], it is shown that such logics are exactly those with a *discriminator term*, whereas in [5], it is established that for an algebraizable logic

[1] For $n \in \omega$ and $\varphi \in Fm_{\mathcal{L}}$, we define $(\neg\sim)^n \varphi$ recursively by $(\neg\sim)^0 \varphi \coloneqq \varphi$ and $(\neg\sim)^{n+1} \varphi \coloneqq \neg\big(\sim(\neg\sim)^n \varphi\big)$.

(in particular, for all bi-intermediate logics), having a discriminator term is a sufficient condition for projectivity.

In order to prove the converse, i.e., that any projective bi-intermediate logic $\vdash$ must contain a theorem of the form $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$, we introduce for each $n \in \omega$ the formula

$$\theta_n := (\neg\sim)^{n+2} p \to (\neg\sim)^{2n+4} \neg(\neg\sim)^n \neg p,$$

and prove that $\vdash \theta_n$ forces $\vdash (\neg\sim)^{2n+3} p \to (\neg\sim)^{2n+4} p$. We then assume that $\vdash$ is projective, so the fact that the formula $\varphi := p \to \neg\sim p$ is unifiable in $\vdash$ (simply take a substitution that sends p to $\top$) entails that it must have a projective unifier σ. It follows that $\varphi \vdash p \leftrightarrow \sigma(p)$. By using the Deduction Theorem for bi-intermediate logics [6], which states that

$$\Gamma, \phi \vdash \psi \iff \exists n \in \omega \left(\Gamma \vdash (\neg\sim)^n \phi \to \psi\right),$$

for every $\Gamma \cup \{\phi, \psi\} \subseteq Fm_{\mathcal{L}}$, we infer that $\vdash (\neg\sim)^n \varphi \to \big(p \leftrightarrow \sigma(p)\big)$, for some $n \in \omega$. Then, with a view to contradiction, we assume that $\nvdash (\neg\sim)^{2n+3} p \to (\neg\sim)^{2n+4} p$, hence $\nvdash \theta_n$ by above. After some semantical combinatorics, the aforementioned consequence of the Deduction Theorem, together with $\nvdash \theta_n$, is enough to arrive at the desired contradiction.

We also showed that bi-IPC is not unitary, by proving that while the formula $p \to \neg\sim p$ is unifiable in bi-IPC, it does not admit a most general unifier.

3 Structural Completeness

Let $\mathcal{L}_0$ be a sublanguage of the bi-intuitionistic language $\mathcal{L}$ and $\vdash$ a logic in $\mathcal{L}_0$. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_0}$ is finite. A rule $\Gamma \rhd \varphi$ is said to be *valid* in $\vdash$ if $\Gamma \vdash \varphi$, and *admissible* in $\vdash$ if for every substitution σ we have that $\vdash \sigma[\Gamma]$ implies $\vdash \sigma(\varphi)$ (that is, if a substitution σ is a $\vdash$-unifier of all the formulas in Γ, then it must also be a $\vdash$-unifier of φ). We denote by $\vdash + \Gamma \rhd \varphi$ the least (wrt. inclusion) logic in $\mathcal{L}_0$ containing $\vdash \cup \langle \Gamma, \varphi \rangle$. The logic $\vdash$ is said to be *structurally complete* if every admissible rule is also valid (the converse holds in general, by substitution invariance). We denote the least (wrt. inclusion) structurally complete logic in $\mathcal{L}_0$ containing $\vdash$ by $Sc(\vdash)$, and call it the *structural completeness* of $\vdash$.

Let bi-IPC$^-$ be the $(\wedge, \vee, \neg, \sim)$-fragment of bi-IPC. A standard and straightforward argument shows that this is the logic induced by the class of matrices $\{\langle \mathbf{A}, \{1\} \rangle \colon \mathbf{A} \in \text{bi-PDL}\}$, where bi-PDL denotes the variety of *double pseudocomplemented distributive lattices*. Notably, this class of algebras enjoys a restricted version of the celebrated *Priestley duality* that associates to each $\mathbf{A} \in$ bi-PDL its *dual bi-p-space* $\mathbf{A}_*$ (see, e.g., [4]). And conversely, to each bi-p-space $\mathcal{X}$ we associate its *double pseudocomplemented dual* $\mathcal{X}^*$. Using this duality, one can show that the class $Mod^*(\text{bi-IPC}^-)$ of reduced matrix models of bi-IPC$^-$ satisfies the equality

$$Mod^*(\text{bi-IPC}^-) = \{\langle \mathbf{A}, \{1\} \rangle \colon \mathbf{A} \in \text{bi-PDL and } dp(\mathbf{A}_*) \leq 2\},$$

where $dp(\mathbf{A}_*)$ denotes the *depth* of the underlying poset of $\mathbf{A}_*$.

In the second part of this talk, I will present an unpublished joint work with Tommaso Moraschini, where we proved that, except for the classical propositional calculus CPC, no consistent and locally finite axiomatic extension of bi-IPC$^-$ is structurally complete[2]. This result is in sharp contrast with [7], where it is shown that every axiomatic extension of the $(\wedge, \vee, \neg)$-fragment of IPC must be structurally complete.

[2] In [1], it is proved that apart from CPC, every bi-intermediate logic is not structurally complete. However, structural completeness results are very sensitive to changes in signature. Moreover, our methods diverge significantly, because unlike all the bi-intermediate logics, the axiomatic extensions of bi-IPC$^-$ are not algebraizable.

Let $\vdash$ be a fixed but arbitrary locally finite axiomatic extension of $\mathsf{bi\text{-}IPC}^-$. If $\mathcal{X}$ is a bi-p-space, we write $\mathcal{X} \in Mod(\vdash)$ when $\langle \mathcal{X}^*, \{1\}\rangle \in Mod(\vdash)$ holds true. Using the equality in the previous display, we prove that the class $FgMod^*(\vdash)_{RSI}$ of finitely generated relatively subdirectly irreducible reduced matrix models of $\vdash$ can be identified with

$$\{\mathbf{A} \in \mathsf{bi\text{-}PDL} \colon \mathbf{A}_* \in Mod(\vdash) \text{ and } \mathbf{A}_* \text{ is a finite connected poset of depth} \leq 2\}.$$

We then interpret [8, Thm. 2.12] within our setting. This result establishes equivalent conditions for a rule to be admissible in a logic over an arbitrary algebraic language. By making use of the properties of bi-p-spaces and their morphisms, and the fact that $\vdash$ was assumed to be locally finite, we derive from the aforementioned interpretation many equivalent conditions for a rule to be admissible in $\vdash$, culminating in the following[3]: the rule $\Gamma \rhd \varphi$ is admissible in $\vdash$ iff for every finite connected poset $\mathcal{X}$ of depth ≤ 2 such that $\mathcal{X} \in Mod(\vdash)$, there exists $\mathcal{Y}$, a finite poset of depth ≤ 2 satisfying $\mathcal{Y} \in Mod(\vdash + \Gamma \rhd \varphi)$, and such that $\mathcal{X}^*$ is a homomorphic image of $\mathcal{Y}^*$. Finally, we use the previous equivalence to show that $Sc(\vdash)$, the structural completeness of $\vdash$, coincides with $Log(\mathbf{K}_\vdash)$, the logic induced by the class of matrices

$$\mathbf{K}_\vdash := \{\langle(\mathcal{X} \uplus \bullet)^*, \{1\}\rangle \colon \mathcal{X} \in Mod(\vdash) \text{ and } \mathcal{X} \text{ is a finite connected poset of depth} \leq 2\},$$

where $\mathcal{X} \uplus \bullet$ denotes the disjoint union of a poset $\mathcal{X}$ with a singleton poset. Then, a semantical argument ensures that if $FgMod^*(\vdash)_{RSI}$ contains a matrix $\langle \mathbf{A}, \{1\}\rangle$ such that the dual bi-p-space $\mathbf{A}_*$ is not a singleton (which is the case for every consistent axiomatic extension of $\mathsf{bi\text{-}IPC}^-$ distinct from CPC), we have that $\vdash \subsetneq Log(\mathbf{K}_\vdash) = Sc(\vdash)$, i.e, that $\vdash$ is not structurally complete. We are currently working on proving the analogous result for arbitrary (i.e., not necessarily locally finite) axiomatic extensions of $\mathsf{bi\text{-}IPC}^-$.

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[3] Note that any finite poset $\mathcal{X}$ can be viewed as a bi-p-space when equipped it with the discrete topology.