Conditionals as quotients in Boolean algebras

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In this contribution we introduce and study a logico-algebraic notion of conditional operator. A conditional statement is a hypothetical proposition of the form

"If [antecedent] is the case, then [consequent] is the case",

where the antecedent is assumed to be true. Such a notion can be formalized by expanding the language of classical logic by a binary operator a/b that reads as "a given b".

A most well-known approach in this direction comes from a philosophical perspective developed first by Stalnaker [9, 10], and further analyzed by Lewis [5], that in order to axiomatize the operator / ground their investigation on particular Kripke-like structures. In particular, Lewis defines a hierarchy of logics for conditionals, which have been shown to be algebraizable in [7] with respect to varieties of Boolean algebras with operators, named *Lewis variably strict conditional algebras* or *V-algebras*.

The novel approach we propose here is grounded in the algebraic setting of Boolean algebras, where we show that there is a natural way of formalizing conditional statements.

The algebraic intuition

Given a Boolean algebra $\mathbf{B}$ and an element b in B, one can define a new Boolean algebra, say $\mathbf{B}/b$, intuitively obtained by assuming that b is true. More in details, one considers the congruence collapsing b and the truth constant 1, and then $\mathbf{B}/b$ is the corresponding quotient. Then the idea is to define a conditional operator / such that a/b represents the element a as seen in the quotient $\mathbf{B}/b$, mapped back to $\mathbf{B}$. The particular structural properties of Boolean algebras allow us to do so in a natural way.

Note that if $b \neq 0$ the quotient $\mathbf{B}/b$ is actually a *retract* of $\mathbf{B}$, which means that if we call π_b the natural epimorphism $\pi_b : \mathbf{B} \to \mathbf{B}/b$, there is an injective homomorphism $\iota_b : \mathbf{B}/b \to \mathbf{B}$ such that $\pi_b \circ \iota_b$ is the identity map. The idea is then to consider

$$a/b := \iota_b \circ \pi_b(a). \tag{1}$$

We observe that the map ι_b is not uniquely determined, meaning that there can be different injective homomorphisms ι, ι' such that $\pi_b \circ \iota = \pi_b \circ \iota'$ is the identity; distinct choices yield distinct values for a/b.

Now, in order to be able to define an operator / over the algebra $\mathbf{B}$, one needs to consider all the different quotients, determined by all choices of elements $b \in B$. Then, if $0 \neq b \leq c$, by general algebraic arguments one gets a natural way of looking at nested conditionals; indeed it holds that $(\mathbf{B}/c)/\pi_c(b) = \mathbf{B}/b$, which means that $\mathbf{B}/b$ is a quotient of $\mathbf{B}/c$, and actually also its retract. It is then natural to ask that the choices for ι_b and ι_c be *compatible*, in the sense that there is a way of choosing the embedding $\iota_{\pi_c(b)}$ so that

$$\iota_b = \iota_c \circ \iota_{\pi_c(b)}, \tag{2}$$

which yields in particular that $a/b = (a/b)/c$ whenever $b \leq c$.

The case where $b = 0$ needs to be considered separately, since the associated quotient is the trivial algebra that cannot be embedded back into $\mathbf{B}$. Since intuitively we are considering the quotients by an element b to mean that "b is true", the *ex falso quodlibet* suggests that we map all elements to 1, i.e:

$$a/0 := 1. \tag{3}$$

The idea is then to use Stone duality to translate the above conditions to the dual setting; in other words, we generate the intended models as algebras of sets.

The standard models via Stone duality

For simplicity, let us describe our setup restricting to finite algebras. By the finite version of Stone duality, we now see the algebra $\mathbf{B}$ as an algebra of sets, say that $\mathbf{B} = \mathcal{S}(X)$ for a set X. Then the above reasoning translates to the following.

Given $Y \subseteq X$, the natural epimorphism $\pi_Y : \mathcal{S}(X) \to \mathcal{S}(Y)$ dualizes to the identity map $\mathrm{id}_Y : Y \to X$, and the embedding $\iota_Y : \mathcal{S}(Y) \to \mathcal{S}(X)$ dualizes to a surjective map $f_Y : X \to Y$, such that $f_Y \circ \mathrm{id}_Y = \mathrm{id}_Y$; in other words, we are asking that f_Y restricted to Y is the identity. Moreover, consider $Y \subseteq Z \subseteq X$. Then the compatibility condition (2) becomes on the dual $f_Y = f^Z_Y \circ f_Z$, where f^Z_Y is the dual of the map $\iota_{\pi_Z(Y)}$. The standard models are those that originate by the above postulates; let us be more precise.

Definition 1. Given a set X, we say that a class of surjective functions $\mathcal{F} = \{f^Z_Y : Z \to Y : \emptyset \neq Y \subseteq Z \subseteq X\}$ with $f^Z_Y : Z \to Y$ is *compatible with* X if:

1. f^X_Y restricted to Y is the identity on Y;
2. $f^X_Y = f^Z_Y \circ f^X_Z$.

We now define the class of standard models as algebras of sets.

Definition 2. A *standard model* is an algebra with operations $\{\wedge, \vee, \neg, /, 0, 1\}$ that is a Boolean algebra of sets $\mathcal{S}(X)$ for some set X with $/$ defined as follows from a class of functions $\mathcal{F}$ compatible with X:

$$Y/Z := (f^X_Z)^{-1}(Y \cap Z)$$

for any $Y, Z \subseteq X$ and $Z \neq \emptyset$, and for any $Y \subseteq X$ we set $Y/\emptyset := X$.

The variety of Quonditional Algebras

Let us call the variety of algebras generated by the standard models QA, and the algebras therein Quonditional Algebras.

In this work, we prove that QA is a subvariety of the variety of Lewis variably strict conditional algebras VA. In particular, with respect to VA one needs to add the algebraic identities characterizing the models of Stalnaker's logic of conditionals [5, 9, 7]:

$$x \wedge y \ \leq \ y/x \ \leq \ x \to y; \quad y/x \ \vee \ \neg y/x = 1,$$

plus the algebraic version of a condition called *uniformity* in [5], and the axiom arising from the above compatibility condition (2):

$$x/y = (x/y)/(y \vee z).$$

Interestingly, this last axiom, which here arises from a purely algebraic compatibility condition, has been discussed in the literature through the name of *flattening* in [1]; there it is used to axiomatize a logic of conditionals introduced in [3], which introduced a particularly simple special case of ordering semantics for conditionals based on functions from the natural numbers to the set of possible worlds.

We then provide an algebraic study of this variety, which in particular turns out to be a *discriminator variety*. In a variety of algebras with a Boolean reduct like in this case, this means that the variety is generated by algebras with a unary term t such that

$$t(0) = 0 \quad \text{and} \quad t(x) = 1, \text{ if } x \neq 0;$$

specifically, we show that all standard models have as discriminator term $t(x) = \neg(0/x)$.

The fact that QA is a discriminator variety, in particular entails that the classes of subdirectly irreducible, directly indecomposable, and simple algebras in QA coincide, and in this case they are exactly the class of (isomorphic copies of) standard models.

Finally, we study the duality theory for QA, in terms of Stone spaces with a ternary relation. With this respect, we observe that the binary operator / is not simply an application of Jónsson-Tarski duality for Boolean algebras with (modal) operators [4]; indeed, the models are not Boolean algebras with an operator in the usual sense, since / is not additive on both arguments (more precisely, it only distributes over meets on the consequent) and it cannot be recovered from a unary modal operator. In particular, we base this analysis on the duality for VA presented in [8].

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