On Non-Classical Polyadic Algebras

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Algebraization of logic has been widely studied by logicians ever since G. Boole discovered the connection between classical propositional logic and two-element Boolean-type algebras. Afterwards, A. Mostowski, A. Tarski, and P. Halmos developed the lattice-based [6], cylindric [4], and polyadic [3] algebraization of classical quantified logic, respectively. To further generalize these ideas, researchers have explored the algebraization of nonclassical quantified logics, leading to the development of structures such as polyadic MV-algebras [8], polyadic BL-algebras [2], polyadic Rasiowa-implicative algebras [7] and cylindric Heyting algebras [5].

Following this line of research, we first define polyadic algebras over algebraically-implicative logics [1]. After constructing functional polyadic algebras, we prove the functional representation theorem, which encompasses many known results for non-classical polyadic algebras.

Let's first fix some notations. Give two sets I, J with $J \subseteq I$. We call a mapping $\sigma : I \to I$ a transformation of I and denote the identity transformation by ι. For $\sigma, \tau \in I^I$, $\sigma J \tau$ means that $\sigma(i) = \tau(i)$ for all $i \in J$. That is, σ and τ agrees on J. Also, we denote $\sigma(I \setminus J)\tau$ as $\sigma J_* \tau$, i.e. σ and τ agree on the complement of I. If $\sigma J_* \iota$, we say J *supports* σ.

Let $\mathcal{L}_{\forall\exists} = \langle \mathcal{O}, \forall, \exists, \mathbf{P}, \mathbf{F}, Var, \rho\rangle$ be a first-order language where $\{\to\} \subseteq \mathcal{O}$ is a set of propositional connectives, $\mathbf{P}(\mathbf{F})$ is a set of relation (functional) symbols, Var is a set of variables, and $\rho : \mathcal{O} \to \omega$ is an arity function.

Similar to classical polyadic algebra developed by Halmos in [3], we first define polyadic $\langle \mathcal{L}_{\forall\exists}, I\rangle$- algebra $\mathbf{A}$ is as

$$\langle A, (\circ^{\mathbf{A}} : \circ \in \mathcal{O}), \forall^{\mathbf{A}}, \exists^{\mathbf{A}}, S^{\mathbf{A}}\rangle$$

where $\circ^{\mathbf{A}} : A^n \to A$ if $\rho(\circ) = n$, $\forall^{\mathbf{A}}, \exists^{\mathbf{A}} : \mathcal{P}_\omega(I) \to A^A$, and $S^{\mathbf{A}} : I^I \to A^A$ such that the following axioms are satisfied :

- $S^{\mathbf{A}}_\iota x = x$;
- $S^{\mathbf{A}}_\sigma(S^{\mathbf{A}}_\tau x) = S^{\mathbf{A}}_{\sigma\tau} x$, for all $\sigma, \tau \in I^I$;
- $S^{\mathbf{A}}_\sigma(\circ^{\mathbf{A}}(x_1, \dots, x_{\rho(\circ)})) = \circ^{\mathbf{A}}(S^{\mathbf{A}}_\sigma x_1, \dots, S^{\mathbf{A}}_\sigma x_n)$, for all $\circ \in \mathcal{O}$, $\sigma \in I^I$;
- $S^{\mathbf{A}}_\sigma Q^{\mathbf{A}}_J x = S^{\mathbf{A}}_\tau Q^{\mathbf{A}}_J x$ for all $Q \in \{\forall, \exists\}$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that $\sigma J_* \tau$;
- $Q^{\mathbf{A}}_J S^{\mathbf{A}}_\sigma x = S^{\mathbf{A}}_\sigma Q^{\mathbf{A}}_{\sigma^{-1}(J)} x$ for all $Q \in \{\forall, \exists\}$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that σ is injective on $\sigma^{-1}(J)$.

We then denote L as algebraically-implicative predicate logic with the language $\mathcal{L}_{\forall\exists}$ as in [1]. By lemma 2.9.11 in [1], $\mathbf{A}$ is an algebra of truth values for L, or an L-algebra, if there is a set of equations $\mathcal{E}$ such that the following quasi-equations hold in $\mathbf{A}$ for each $\alpha \approx \beta \in \mathcal{E}$:

- $\alpha(\varphi) \approx \beta(\varphi)$, for each axiom φ of L
- $\bigwedge \mathcal{E}[\Gamma] \Rightarrow \alpha(\varphi) \approx \beta(\varphi)$ for each rule $\Gamma \vdash_{\mathrm{L}} \varphi$ of L

- $\bigwedge \mathcal{E}[x \leftrightarrow y] \Rightarrow x \approx y$

Then we define that a polyadic $\langle \mathcal{L}_{\forall\exists}, I\rangle$-algebra $\mathbf{A}$ is called a polyadic L-algebra if it satisfies the following equations and quasi-equations :

- Axioms of L-algebras;

- Axioms (T1)-(T8) for all $\sigma \in I^I$ and $J \subseteq_\omega I$ as in [7].

On the other hands, following the definition in [7], we say a value $\mathcal{L}_{\forall\exists}$-algebra $\mathbf{V}$ is an algebra of the form

$$\langle V, (\circ^{\mathbf{V}} : \circ \in \mathcal{O}), \forall^{\mathbf{V}}, \exists^{\mathbf{V}}\rangle$$

where $\circ^{\mathbf{V}} : V^{\rho(\circ)} \to V$ is a $\rho(\circ)$-ary operation on V for each $\circ \in \mathcal{O}$, and $Q^{\mathbf{V}} : \mathcal{P}(V) \rightharpoonup V$ is a partial unary second-order operation with domain on power set $\mathcal{P}(V)$ of V for each $Q \in \{\forall, \exists\}$.

Therefore, given a value $\mathcal{L}_{\forall\exists}$-algebra $\mathbf{V}$ and two sets X, I. A *functional polyadic* $\langle \mathcal{L}, I\rangle$-*algebra* $\bar{\mathbf{V}}$ is of the form

$$\langle V^{X^I}, (\circ^{\bar{\mathbf{V}}} : \circ \in \mathcal{O}), \forall^{\bar{\mathbf{V}}}, \exists^{\bar{\mathbf{V}}}, S^{\bar{\mathbf{V}}}\rangle$$

where $\circ^{\bar{\mathbf{V}}} : (V^{X^I})^{\rho(\circ)} \to V^{X^I}$, $\forall^{\bar{\mathbf{V}}}, \exists^{\bar{\mathbf{V}}} : \mathcal{P}_\omega(I) \to [V^{X^I}, V^{X^I}]$, and $S^{\bar{\mathbf{V}}} : I^I \to End(\mathbf{V})$ are defined as follows :

- $(\circ^{\bar{\mathbf{V}}}(p_1, \ldots, p_{\rho(\circ)}))(\vec{x}) = \circ^{\mathbf{V}}(p_1(\vec{x}), \ldots, p_{\rho(\circ)}(\vec{x}))$ for all $p_1, \ldots, p_{\rho(\circ)} \in V^{X^I}$ and $\vec{x} \in X^I$;

- $(\forall^{\bar{\mathbf{V}}}_J p)(\vec{x}) = \forall^{\mathbf{V}}(\{p(\vec{y}) : \vec{x} J_* \vec{y}\})$, for all $p \in V^{X^I}$, $J \subseteq_\omega I$, and $\vec{x}, \vec{y} \in X^I$; similarly for $\exists^{\bar{\mathbf{V}}}$

- $(S^{\bar{\mathbf{V}}}_\sigma p)(\vec{x}) = p(\sigma_* x)$ where $(\sigma_* \vec{x})_i = (\vec{x})_{\sigma(i)}$ for all $\sigma \in I^I$ and $\vec{x} \in X^I$.

Note that we use $[V^{X^I}, V^{X^I}]$ to denote that $\forall^{\bar{\mathbf{V}}}_J p$ and $\exists^{\bar{\mathbf{V}}}_J$ are total functions from X^I to $\mathbf{V}$. If $\langle V, (\circ^{\mathbf{V}} : \circ \in \mathcal{O})\rangle \in \mathbf{ALG}^*(\mathrm{L})$, the algebra of reduced models of L, and $\forall^{\mathbf{V}}$ and $\exists^{\mathbf{V}}$ are respectively the generalized meet and join operations, then we say $\bar{\mathbf{V}}$ is a functional polyadic L-algebra. We can prove a similar theorem as in [3] :

Theorem 1. *Every functional polyadic* L-*algebra is a polyadic* L-*algebra.*

To see the connection with algebraically-implicative predicate logic, let $\mathfrak{M}$ be a reduced model for L and $P_{\mathfrak{M}}$ is the interpretation of predicate symbols $P \in \mathbf{P}$ in $\mathfrak{M}$. We can show the following lemma.

Lemma 1. *Let $\mathcal{F}(\mathfrak{M})$ be a subalgebra of $\bar{\mathbf{A}}$ (with $X = M$ and $I = Var$) generated by $\{P_{\mathfrak{M}} \mid P \in \mathbf{P}\}$. Then $\mathcal{F}(\mathfrak{M})$ is a functional polyadic $\langle \mathcal{L}_{\forall\exists}, Var\rangle$-algebra.*

To prove the converse case, it's similar to the classical case that we need to impose some further constrain on the polyadic algebras. We say an element a of a polyadic $\langle \mathcal{L}_{\forall\exists}, I\rangle$-algebra has a finite support $J \subseteq I$ if $S_\sigma a = S_\tau a$ for all $\sigma, \tau \in I^I$ such that $\sigma J \tau$. A polyadic $\langle \mathcal{L}_{\forall\exists}, I\rangle$-algebra is locally finite if every element has a finite support. Hence, we can prove the following functional representation theorem.

Theorem 2. *Every locally finite polyadic* L-*algebra of infinite dimension is isomorphic to a functional polyadic* L-*algebra.*

As a case study, we investigate the algebraization of first-order relevant logics. Let $\mathcal{L}_{RQ} = \langle \{\wedge, \vee, \sim, \circ, 1, \to\}, Con, Pred, \forall, I, \rho\rangle$ where Con is a set of name constant symbols (i.e. 0-ary functional symbols), $Pred$ is a set of predicate symbols of varying arities, I is a countable set of variables, and ρ is an arity function. A *polyadic* $\langle \mathcal{L}_{RQ}, I\rangle$*-De Morgan Monoid* is an algebra of the form:

$$A := \langle A; \wedge, \vee, \sim, \circ, \to, 1, \langle \forall^A_J \mid J \subseteq_\omega I\rangle, \langle S^A_\sigma \mid \sigma \in I^{(I)}\rangle\rangle$$

that satisfies the following axioms:

(Poly) Axioms of polyadic $\langle \mathcal{L}_\forall, I\rangle$-algebras

(DMM) The defining equations of De Morgan Monoids

(Q1) $\forall_J 1 = 1$;

(Q2) $\forall_J p \leq p$;

(Q3) $\forall_J(p \wedge q) = \forall_J p \wedge \forall_J q$;

(Q4) $\forall_J \forall_J p = \forall_J p = \sim \forall_J \sim \forall_J p$;

(Q5) $\forall_J(p \to q) \leq (\forall_J p \to \forall_J q)$;

(Q6) $\forall_J(\forall_J p \to \forall_J q) = \forall_J p \to \forall_J q$;

(Q7) $\forall_J(p \vee q) = \sim \forall_J \sim p \vee \forall_J q$.

We can construct functional polyadic De Morgan Monoids similarly. Therefore, we have the following theorem.

Theorem 3. *Every functional polyadic* $\langle \mathcal{L}_{RQ}, I\rangle$*-De Morgan Monoid is a polyadic* $\langle \mathcal{L}_{RQ}, Var\rangle$*-De Morgan Monoid.*

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