Tropicalization through the lens of Łukasiewicz logic, with a topos theoretic perspective

Antonio Di Nola[1], Brunella Gerla[2], and Giacomo Lenzi[3] (corresponding)

[1] Department of Mathematics
University of Salerno
adinola@unisa.it
[2] Department of Theoretical and Applied Sciences
University of Insubria
brunella.gerla@uninsubria.it
[3] Department of Mathematics
University of Salerno
gilenzi@unisa.it

The main aim of this paper is to show that the topics of Łukasiewicz logic, semirings and tropical structures fruitfully meet, by combining the ideas of [1], [6], [15] and [4]. This gives rise to a topos theoretic perspective to Łukasiewicz logic, see [2]. Our aim seems to be completely in line with the spirit of the following remark from [12]:

> "*The confluence of geometric, combinatorial and*
> *logical-algebraic techniques on a common problem*
> *is one of the manifestations of the unity of mathematics.*"

Tropicalization, originally, is a method aimed at simplifying algebraic geometry and it is used in many applications. It can be seen as a part of idempotent algebraic geometry. Tropical geometry produces objects which are combinatorially similar to algebraic varieties, but piecewise linear. That is, tropicalization attaches a polyhedral complex to an algebraic variety, obtaining a kind of algebraic geometry over idempotent semifields.

Originally the idea was focused on complex algebraic geometry and started from a field (or ring) K and its polynomials, an idempotent semiring $(S, \wedge, +, 0, 1)$ and a valuation $v : K \to S$. A semiring is an abelian monoid $(A, +, 0)$ with a multiplicative monoid structure $(A, \cdot, 1)$ satisfying the distributive laws and such that $a \cdot 0 = 0 \cdot a = 0$ for all $a \in A$. A semiring is idempotent if $+$ is idempotent.

Usually $S = G \cup \{\infty\}$ where G is a totally ordered abelian group and ∞ is an infinite extra element, but we suggest to be more liberally inspired by [15]: first, G can be any lattice ordered abelian group, and $S = G \cup \{\infty\}$ is a semifield (the *Rump semifield* of G) . Moreover, we point out that the passage from G to S and conversely is functorial, even an equivalence. So we propose here a *functorial tropicalization*.

A polynomial p with coefficients in K is replaced by a polynomial $Trop(p)$ with coefficients in S. In the tropicalization $Trop(p)$ of p, the polynomial product of K becomes $+$ in S, the polynomial sum in K becomes the infimum in S. The constants $k \in K$ become their value $v(k)$. The idea is that the tropicalized polynomial $Trop(p)$ should be simpler than p but retain the combinatorial structure of p.

In this paper we intend to contribute to what we mean by tropical mathematics, which for us, in a very broad sense, is the use of idempotent semirings in mathematics.

The most used semiring in tropicalization is the tropical semiring

$$\mathbb{R}^{maxplus} = (\mathbb{R} \cup \{-\infty\}, max, +, -\infty, 0)$$

(or its dual with min instead of max). It is used in many contexts, from algebraic geometry to optimization to phylogenesis. An application to differential equations is given in [8]. The variety generated by $\mathbb{R}^{maxplus}$ includes the semiring reduct of $[0,1]$, but not the other way round. Instead, if we consider the negative cone of the reals, $\mathbb{R}^{neg} = (\mathbb{R}^{\leq 0} \cup \{-\infty\}, max, +, -\infty, 0)$, we have that the varieties generated by $[0,1]$ and $\mathbb{R}^{neg}$ are the same. Another famous semiring is the tropical semiring $\mathbb{N}^{trop} = (\mathbb{N} \cup \{+\infty\}, min, +, +\infty, 0)$. Once again, the varieties generated by $\mathbb{N}^{trop}$ and $[0,1]$ are the same. However, $\mathbb{N}^{trop}$ and $[0,1]$ have different first order theories. Thus, looking at these two varieties from a logical point of view, some differences emerge. This justifies approaching tropical algebraic structures from a logical point of view.

One can try to apply the same tropicalization idea to more general frameworks, like universal algebraic geometry by Plotkin [13]. This paper proposes a possible generalization of tropicalization of algebraic structures using many valued logic and especially fuzzy logic. In fact, the algebraic structures of fuzzy Łukasiewicz logic (MV-algebras) have a natural structure of idempotent semiring. Note that MV-semirings are mentioned in [7] in the context of weighted logics.

Łukasiewicz logic is a fuzzy logic, that is a logic where the set of values is the real interval $[0,1]$, rather than the case $\{0,1\}$ of classical logic. The connectives of Łukasiewicz logic are $x \oplus y = min(x+y, 1)$ (replacing OR) and $\neg x = 1 - x$ (replacing NOT). Then AND gets replaced by $x \odot y = max(0, x+y-1)$ (the Łukasiewicz product). We have the tertium non datur, $x \oplus \neg x = 1$, the non contradiction law $x \odot \neg x = 0$, but not the idempotency: $x \oplus x \neq x$. Despite the lack of idempotency, MV-algebras are deeply connected with idempotent structures.

Łukasiewicz logic can be axiomatized as follows, where $x \to y$ means $\neg x \oplus y$:

1. $x \to (y \to x)$
2. $(x \to y) \to ((y \to z) \to (x \to z))$
3. $((x \to y) \to y)) \to ((y \to x) \to x)$
4. $(\neg x \to \neg y) \to (y \to x)$.

The only rule is modus ponens: from $x \to y$ and x derive y.

Like the semantic counterpart of classical logic is given by Boolean algebras, the semantic counterpart of Łukasiewicz logic is given by MV-algebras. These are algebraic structures generalizing Boolean algebras and widely used in many applications, from quantum mechanics to games to functional analysis. The interplay between semirings and MV-algebras is very interesting. Every MV-algebra is in a natural way (better, in two natural dual ways) a semiring, as explained in [5], [4].

Namely, MV-algebras are structures $(A, \oplus, 0, 1, \neg)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg 0 = 1$, $x \oplus 1 = 1$, $\neg\neg x = x$ and $\neg(\neg x \oplus y) \oplus y) = \neg(\neg y \oplus x) \oplus x)$.

MV-algebras are also lattices under the ordering $x \leq y$ such that there is z with $y = x \oplus z$. We let also $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \ominus y = x \odot \neg y = \neg(\neg x \oplus y)$.

The main example of MV-algebra is $[0,1]$ with $x \oplus y = min(x+y, 1)$ and $\neg x = 1 - x$. By [3] it follows that $[0,1]$ generates the variety of MV-algebras.

The semiring reducts of an MV-algebra A are $(A, 0, 1, \wedge, \oplus)$ and $(A, 0, 1, \vee, \odot)$. They are isomorphic semirings and an isomorphism is given by the negation (see [4]).

In order to define other interesting MV-algebras it is convienient to introduce Mundici functor Γ, see [9]. Γ is a functorial equivalence between MV-algebras and Abelian ℓ-groups with strong unit. This equivalence was originally used in order to clarify the relations between

AF C^*-algebras and their K_0 group. $\Gamma(G,u)$ is the interval $[0,u]$ of G where $x \oplus y = (x+y) \wedge u$ and $\neg x = u - x$.

Another important example is the Chang MV-algebra $C = \Gamma(Z \overset{lex}{\times} Z,\ (1,\ 0))$ where $\overset{lex}{\times}$ denotes lexicographic product of groups. We call $V(C)$ the variety generated by C and $V(C)$-algebras the elements of $V(C)$. The algebra C has a very special role in MV-algebras theory. $\Gamma^{-1}(C)$ is the K_0 of Behncke-Leptin C^*-algebra $A_{0,1}$, see [10].

An equivalence related to Γ is the one called Δ between abelian ℓ-groups and perfect MV-algebras, see ([6]), that is, MV-algebras generated by their infinitesimals. Namely, if G is an abelian ℓ-group, then $\Delta(G) = \Gamma(Z \overset{lex}{\times} G, (1,0))$. So, C can also be defined as $\Delta(Z)$.

We can think of the logic of perfect MV-algebras as the logic of the quasi true or quasi false.

We focus on algebraic models of an equational extension of Łukasiewicz logic. Actually we consider the extension of Łukasiewicz logic given by the equation in the section 3.1.

It is often stressed that enriched valuations represent "perturbations" of valuations of classical models. In our case we can speak of "infinitesimal perturbation" of classical propositional logic, that in turn, means infinitesimal perturbation of Boolean algebras. Although it seems rather exotic to consider infinitesimally perturbed Boolean algebras as models of a propositional logic [11], it happens that these models can have an elegant algebraic characterization. Indeed, we can see that, albeit by means of a categorical equivalence, such models can be described by a Stone space and a family of lattice ordered abelian groups. Actually we will consider weak boolean products of lattice ordered abelian groups. It will be quite interesting that such a logic may have theories interpreted into points of the presheaf topos over the multiplicative monoid of integers.

As future work, we note that we propose a tropicalization functor that should be compared with [2] and [14]. In [2], theorem 2.1, the authors relate subgroups of the rationals to the points of a natural topos, and in this paper we transfer this correspondence into a result on a category of countable, perfect, linearly ordered MV algebras, via the functor Δ. It should be possible to extend this result to more general categories of MV algebras, using other topoi. In [14] in particular the problem of negation is addressed, which is an issue also for us, since the functor θ gives structures which do not have a natural involutive negation

References

[1] Cignoli, Roberto L. O.; D'Ottaviano, Itala M. L.; Mundici, Daniele, Algebraic foundations of many-valued reasoning. Trends in Logic-Studia Logica Library, 7. Kluwer Academic Publishers, Dordrecht, 2000. x+231 pp.

[2] Alain Connes and Caterina Consani, Geometry of the arithmetic site, Advances in Mathematics, Volume 291, 19 March 2016, Pages 274–329.

[3] Antonio Di Nola, Representation and reticulation by quotients of MV-algebras. Ricerche Mat. 40 (1991), no. 2, 291–297 (1992).

[4] Antonio Di Nola, Brunella Gerla, Algebras of Łukasiewicz's Logic and their Semirings Reducts, Contemp. Math. 377, 131–144, (2005).

[5] Antonio Di Nola, Giacomo Lenzi, On semirings and MV-algebras, proceedings of FUZZ-IEEE 2017, 1–6.

[6] Di Nola, Antonio; Lettieri Ada. Perfect MV-algebras are categorically equivalent to abelian l-groups. Studia Logica 53 (1994), no. 3, 417–432.

[7] Manfred Droste, Paul Gastin, Weighted automata and weighted logics, Theoretical Computer Science, Volume 380, Issues 1-2, 2007, Pages 69-86.

[8] Giansiracusa, Jeffrey, and Stefano Mereta. A general framework for tropical differential equations. Manuscripta mathematica (2023): 1–32.

[9] Mundici, Daniele Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. J. Funct. Anal. 65 (1986), no. 1, 15–63.

[10] Mundici, D., AF algebras with lattice-ordered K_0: Logic and computation, Annals of Pure and Applied Logic 174 (2023) 103182.

[11] Mundici, D., The diffential semantics and syntactic consequence of Łukasiewicz sentential calculus. ArXive:1207.5713v1[math.LO](2012).

[12] Mundici, D., La Logica dei Poliedri, Bollettino dell'Unione Matematica Italiana, Serie 9, (2008).

[13] B. Plotkin, Seven Lectures on the Universal Algebraic Geometry, preprint arXiv:math/0204245.

[14] Rowen, L. H. (2016). Algebras with a negation map. arXiv preprint arXiv:1602.00353.

[15] Rump, Wolfgang, Algebraically closed abelian l-groups. Math. Slovaca 65 (2015), no. 4, 841–862.