Exchangeability and statistical models in non-classical logic

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Probability theory and fuzzy logic are both theories used when one aims at performing some sort of inference in uncertain or imprecise situations. These two theories have been combined in many different ways and for this talk we recall [1], where the authors define the algebraic counterpart of a random variable within a conservative expansion of Łukasiewicz logic. The probabilistic setting used there is the one of subjective probability, as introduced by Bruno de Finetti. Another point of view on probability theory is the so-called frequentist approach. In this case, the probability of an event is defined by the frequency of that event based on previous observations. In this setting, it is common to define a statistical model that fits the observed data, and to derive the properties of the hidden probability distribution in this way. Perhaps surprisingly, subjective probability is related to frequentist probability via de Finetti's work, and in particular via the notion of *exchangeability* that, loosely speaking, shows how statistical models appear in a Bayesian framework, and how probabilities can come from statistics.

In this talk, we mainly discuss the state of the art on the search for a *good* notion of exchangeability and statistical models in non-classical logic, staring with the results of [6]. In particular, we discuss the notion of exchangeability in the setting of Łukasiewicz logic, give a version of the celebrated de Finetti's theorem in algebraic logic, and present the definition and some results on statistical models that are new to the LATD audience.

Classically, exchangeability is defined as follows. Let T be a non-empty set with a σ-algebra of subsets $\mathcal{X}$ and let $\overline{\mathcal{X}}$ be the smallest σ-algebra on T^ω that contains all sets of type $C(E_{i_1},\ldots,E_{i_k}) = \prod_n E_n$ with $E_n = T$ for $n \notin \{i_1,\ldots,i_k\}$. A generic measure σ on $\overline{\mathcal{X}}$ is called *exchangeable* if for any permutation π of $\{i_1,\ldots,i_k\}$,

$$\sigma\left(C(E_{i_1},\ldots,E_{i_k})\right) = \sigma\left(C(E_{\pi(i_1)},\ldots,E_{\pi(i_k)})\right).$$

For any measure μ on $\mathcal{X}$, let $\overline{\mu}$ denote the unique product measure defined on $\overline{\mathcal{X}}$ using μ. A generic measure σ on $\overline{\mathcal{X}}$ is called *presentable* if there exists a measure ν on the set $\mathcal{P}$ of all probability measure on $\mathcal{X}$ such that for any $A \in \overline{\mathcal{X}}$,

$$\sigma(A) = \int_{\mathcal{P}} \overline{\mu}(A) d\nu(\mu).$$

Then, de Finetti's theorem (in the version of Hewitt and Savage [5]) gives sufficient conditions for the two notions to coincide.

To give a non-classical version of this result, the algebraic framework used will be the one of MV-algebras. This is mainly due to the fact that an algebraic theory of probability is encoded in MV-algebras by the notion of *states*. Via a suitable version of the Riesz representation theorem, the so-called Kroupa-Panti theorem, states of MV-algebras are in one-one correspondence with probability measures on a σ-algebra that depends on A.

In particular, we will work with a subclass of the infinitary variety $\mathbf{RMV}_\sigma$ of σ-complete Riesz MV-algebras, that is, MV-algebras closed to multiplication by elements of $[0,1]$ and to countable suprema, see [4]. The subclass needed is the pre-variety generated by $[0,1]$. Algebras in $ISP([0,1])$ are called σ-semisimple and they can be characterized as follows, when they are countably generated.

For a countable cardinal κ, $\mathsf{Borel}([0,1]^\kappa)$ denotes the MV-algebra of $[0,1]$-valued and Borel-measurable functions over the domain $[0,1]^\kappa$. Such algebra is proved to be the free κ-generated algebra in $\mathbf{RMV}_\sigma$ in [2]. If needed, when X is a countable set of generators, we will write $\mathsf{Borel}(X)$ instead of $\mathsf{Borel}([0,1]^{|X|})$. For a topological space T, $\mathcal{BO}(T)$ denotes the σ-algebra of its Borel subsets, that is, the σ-algebra generated by open subsets of T. Now, with a characterization proved in [2], a (at most) countably generated algebra A is σ-semisimple iff there exist sets $\kappa \leq \omega$ and V, with V being an arbitrary intersection of Borel subsets of $[0,1]^\kappa$, such that $A \simeq \mathsf{Borel}([0,1]^\kappa)|_V$, the algebra of restrictions to V of elements of $\mathsf{Borel}([0,1]^\kappa)$.

In this setting, we define an appropriate counterpart of a product measure. To do so, thinking of states as an algebraic dual of probability measures, we will first give a characterization for the coproduct of objects in $\mathbf{RMV}_\sigma$.

Proposition 1. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of σ-semisimple algebras in $\mathbf{RMV}_\sigma$. For any $n \in \mathbb{N}$, let $A_n \simeq \mathsf{Borel}(X_n)|_{V_n}$, where we assume the sets X_n to be countable and pairwise disjoint and we assume each V_n to be a Baire subset of $[0,1]^{X_n}$. Then the free product $\bigoplus_n A_n$ exists in $\mathbf{RMV}_\sigma$ and*

$$\bigoplus_n A_n = \mathsf{Borel}\left(\bigcup_n X_n\right)|_V \qquad \text{with } V = \prod_{n\in\mathbb{N}} V_n.$$

Using this characterization of coproduct, we will define the notion of *presentable state* and *exchangeable state*, and prove that they coincide on any $\mathsf{Borel}([0,1]^\kappa)$, when κ is countable.

Definition 2. Let $A \in \mathbf{RMV}_\sigma$ countably generated, $A \simeq \mathsf{Borel}([0,1]^\kappa)|_V$. Take the coproduct $\bigoplus_\omega A$ of A with itself countably many times. A σ-state $s : \bigoplus_\omega A \to [0,1]$ is called *weakly exchangeable* if the associate measure (in the sense sketched above, by the Kroupa-Panti theorem), on the product of countable copies of $(V, \mathcal{BO}(V))$, is exchangeable in the classical sense. Similarly, the σ-state s is called *weakly presentable* if the associated measure is presentable in the classical sense.

Theorem 3 (Weak de Finetti's exchangeability)**.** *Let κ be a countable cardinal. A state on $\mathsf{Borel}([0,1]^\kappa)$ is weakly exchangeable if, and only if, it is weakly presentable.*

In the second part of the talk we present a logico-algebraic take on the notion of a *statistical model* introduced in [6] and further discussed in [3].

Formally, for $\kappa \leq \omega$, a logico-algebraic statistical model is a function $\eta = (\eta_i)_{i\in\kappa} \colon P \to \Delta_\kappa$, where $P \subseteq [0,1]^d$ is an intersection of Borel measurable sets and Δ_κ is the standard κ-dimensional simplex. When $\kappa = \omega$, we take Δ_ω to be $\{x \in [0,1]^\omega \mid \sum_{i=1}^{\infty} x_i \leq 1\}$, which is known to be closed (and convex) and therefore it is a Borel subset of $[0,1]^\omega$. The intuition behind this definition is the following:

- $[0,1]^\kappa$ is the set of observations on the real world and $\mathsf{Borel}([0,1]^\kappa)$ is the algebra of many-valued events;
- the set $P \subseteq [0,1]^d$ is the set of states of the world, or parameters, we allow d to be any countable cardinal;
- the tuple of functions $\eta := (\eta_i)_{i\in\kappa} \colon P \to [0,1]^\kappa$ is our statistical model: to each parameter $\mathbf{x} \in P$ it associates the tuple $(\eta_i(\mathbf{x}))_{i\in\kappa}$. Each $\eta_i \colon [0,1]^d \to [0,1]$ is a Borel measurable function.

We call κ-dimensional any statistical model whose codomain is Δ_κ.

We will show how κ-dimensional statistical models can be interpreted in the category of σ-complete Riesz MV-algebras and how they can be seen as a suitable pre-sheaf, whose domain is a category of *parameters*.

Lastly, if time permits, we present a (work in progress) approach to exchangeability for states that does not require the Kroupa-Panti theorem and it is related to the more general framework of subjective decision theory.

References

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