Abstract Booklet

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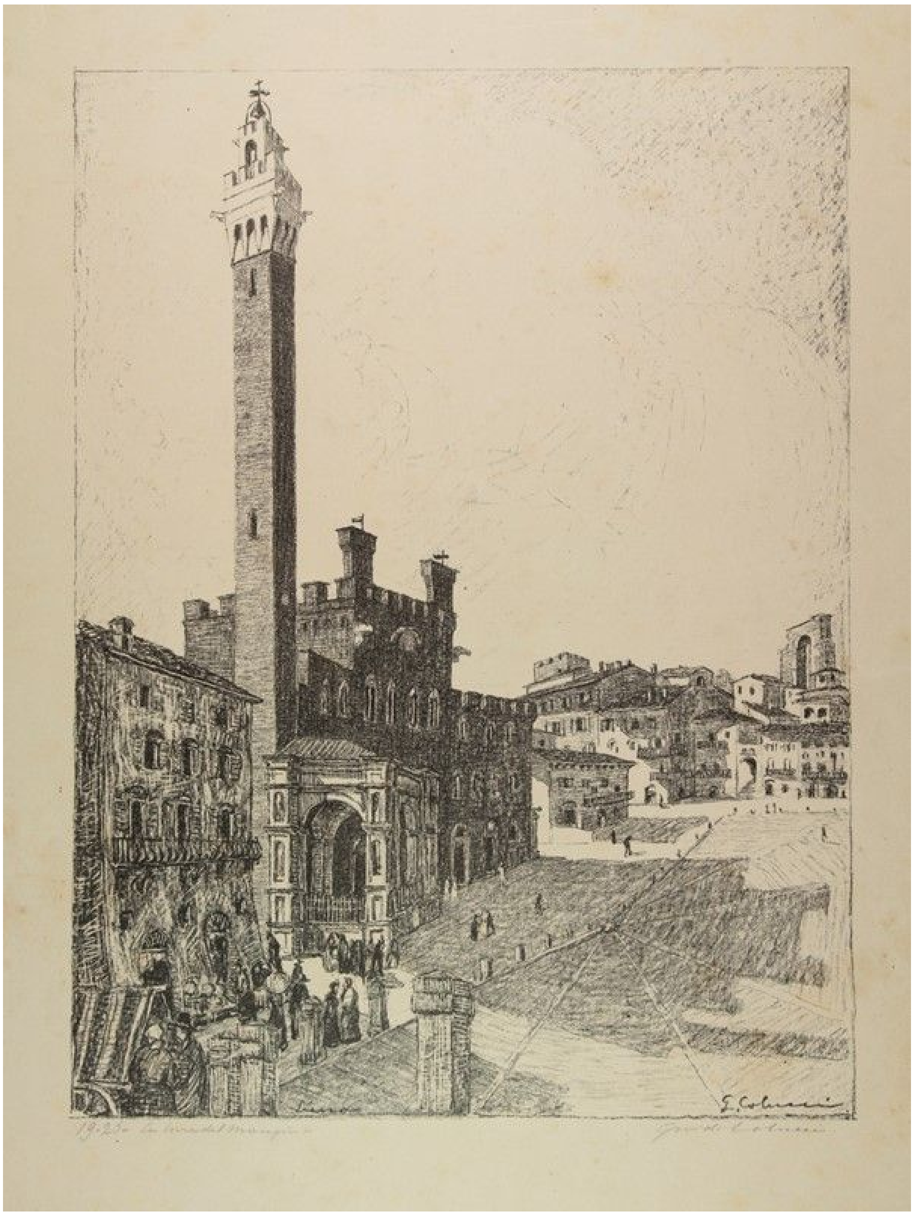

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Contents

Residuated lattices represented by twist-structures

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Residuated lattices arise in many contexts, particularly in algebraic logic, as they provide the algebraic semantics for substructural logics.

The class of residuated lattices is, in itself, a very broad class that includes a wide range of structures of different kinds, many of which are algebraic semantics of well-known and extensively studied propositional logics. Given this great diversity of algebras within the class, the systematic study of residuated lattices often employs constructions to obtain new structures from simpler or better-known ones. In this talk, we will focus on one such constructions, which has become well-known due to the variety of cases it covers: the twist-construction.

Although the first applications of twist-structures were used to obtain lattices with involution (Kalman in 1958), several other authors considered expansions with additional operations which induce new and interesting operations on the twist-structure. In particular, starting from a residuated lattice, the resulting construction yields a new one.

Our aim is to present a unified approach that offers a deeper insight into the classes of residuated lattices that admit a representation based on twist- structures. Our framework encompasses Nelson residuated lattices, Nelson paraconsistent residuated lattices, Kalman residuated lattices, among others. Moreover, we will show that with this approach we can also capture non-involutive twist-structures, such as Quasi-Nelson algebras and some of its variants, which have been recently introduced and studied mainly by U. Rivieccio. Our results enable comparisons among different twist-structures and provide some interesting new examples.

The ideas of presented are based on the works [1] and [2], done in collaboration with N. Galatos, M. Marcos and U. Rivieccio.

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Weighted logics and weighted automata

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Quantitative models and quantitative analysis in Computer Science are receiving increased attention. The goal of this talk is to investigate quantitative automata and quantitative logics. Weighted automata on finite words have already been investigated in seminal work of Schützenberger (1961). They consist of classical finite automata in which the transitions carry weights. These weights may model, e.g., the cost, the consumption of resources, or the reliability or probability of the successful execution of the transitions. This concept soon developed a flourishing theory, as is exemplified and presented in several books by Eilenberg, Salomaa-Soittola, Kuich-Salomaa, Berstel-Reutenauer, Sakarovitch, and the "Handbook of Weighted Automata". We investigate weighted automata and their relationship to weighted logics. For this, we present syntax and semantics of a quantitative logic; the semantics counts 'how often' a formula is true in a given word. Our main result, jointly with Paul Gastin, extending classical results of Büchi, Elgot and Trakhtenbrot (1961), shows that if the weights are taken from an arbitrary semiring, then weighted automata and a syntactically defined fragment of our weighted logic are expressively equivalent. A corresponding result holds for infinite words. Moreover, this extends to quantitative automata investigated by Henzinger et al. for modeling limit average-type or discounting behaviors e.g. of power plants. Finally, we consider Fagin's seminal result (1974) characterizing NP in terms of existential second-order logic; this started the field of descriptive complexity theory. In very recent work, jointly with Guillermo Badia, Carles Noguera and Erik Paul, we obtained a weighted version of Fagin's result.

Interpolation and Amalgamation in Many-Valued Logics

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Fuzzy logics have long enjoyed a fruitful symbiosis with algebraic methods. In this talk, I will discuss some recent successes of this symbiosis in the context of interpolation and amalgamation. We will see both how the advancement of algebraic techniques have enabled rapid progress on interpolation in fuzzy logics, as well as how the challenges posed by fuzzy logics have illuminated the path to new tools for the study of amalgamation in general algebraic systems.

The centerpiece of this discussion will be my recent classification with S. Santschi of varieties of BL-algebras with the amalgamation property, which yields also an exhaustive classification of axiomatic extensions of Hájek's basic fuzzy logic with the deductive interpolation property. I will discuss the main ideas and technical challenges of this classification, and what it can teach us about interpolation and amalgamation writ large.

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[3] W. Fussner and S. Santschi. Interpolation in Hájek's Basic Logic, Ann. Pure. Appl. Logic 176(9), paper no. 103615 (2025).

Conditional logics: a proof-theoretic perspective

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Conditional logics, introduced by David Lewis in 1973, extend classical propositional logic with a binary modal operator which captures fine-grained notions of conditionality, such as counterfactual reasoning or non-monotonic inferences. Analytic proof systems for these logics adapt the methods developed for modal logic, and are defined either by extending the language of sequent calculus through labels or by adding structural connectives, as in nested or hypersequent calculi.

In this talk, I will present sequent calculi for conditional logics that exemplify both approaches: a labelled sequent calculus that modularly captures a wide range of systems, and a nested-style calculus that employs a structural connective corresponding to neighborhoods in the semantic models. These calculi are grounded in neighborhood semantics, which provide a flexible framework for representing conditionals. I will conclude by discussing recent developments on intuitionistic conditional logics, defined by adding the conditional operator to intuitionistic propositional logic, and outline corresponding proof-theoretic systems.

This talk is based on joint works with: Tiziano Dalmonte, Bjoern Lellmann, Sara Negri, Nicola Olivetti and Gian Luca Pozzato.

Kites, pseudo-MV algebras and ℓ-groups

Tomasz Kowalski (joint work with Michal Botur)

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A kite is a certain construction with origins in work of Jipsen and Montagna, later developed by Dvurecenskij and TK, and later yet by Botur and TK. In particular, it yields a categorical equivalence between perfect pseudo-MV algebras and lattice-ordered groups with a distinguished automorphism. Combining several well-known facts about ℓ-groups, residuated lattices and pseudo-MV algebras, this result can be presented as a pure ℓ-group embedding result. Iterating the embedding we can construct an embedding of an arbitrary ℓ-group into an ℓ-group with no nontrivial outer automorphisms. The result is not new, it was known as early as 1973 to McCleary, under the assumption of GCH. In 2000, Droste and Shelah, eliminated the need for GCH. But both McCleary and Droste-Shelah results are in fact about Holland representations: they show that any ℓ-group $Aut(C)$ for a chain C can be embedded into an ℓ-group $Aut(D)$ for a chain D, such that $Inn(Aut(D)) = Aut(Aut(D))$. We show less, namely, that any ℓ-group $\mathbf{G}$ can be embedded into an ℓ-group $\mathbf{H}$ such that $Inn(\mathbf{H}) = Aut(\mathbf{H})$, but we use elementary techniques that we discovered via working with kites and pseudo-MV algebras. Whether McCleary and Droste-Shelah results can be fully recovered by our techniques remains to be seen.

The algebraic structure of spaces of integrable functions

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Abstract

We characterize the functions $\mathbb{R}^I \to \mathbb{R}$ that preserve integrability, meaning that their pointwise application maps any I-tuple of integrable functions to an integrable function (as, for example, the sum $+\colon \mathbb{R}^2 \to \mathbb{R}$). We show that Dedekind σ-complete truncated vector lattices are precisely the algebras with integrability-preserving functions as function symbols and that satisfy all equations true in $\mathbb{R}$. We also show that an analogous study restricted to finite measure spaces gives the class of Dedekind σ-complete vector lattices with weak unit. Furthermore, we provide concrete models for free algebras in these categories.

Introduction

Operations that preserve integrability

We investigate the operations that are somehow implicit in the theory of integration by addressing the following question: which operations preserve integrability, in the sense that they return integrable functions when applied to integrable functions?

To clarify the question, we recall some definitions.

For $(\Omega, \mathscr{F}, \mu)$ a measure space (with the range of μ in $[0, +\infty]$), we write

$$\mathscr{L}^1(\mu) \coloneqq \left\{ f\colon \Omega \to \mathbb{R} \mid f \text{ is } \mathscr{F}\text{-measurable and } \int_\Omega |f| \, d\mu < \infty \right\}.$$

It is well known that, for $f, g \in \mathscr{L}^1(\mu)$, we have $f + g \in \mathscr{L}^1(\mu)$. In other words, $\mathscr{L}^1(\mu)$ is closed under the pointwise addition, induced by the addition of real numbers $+\colon \mathbb{R}^2 \to \mathbb{R}$. More generally, consider a set I and a function $\tau\colon \mathbb{R}^I \to \mathbb{R}$, which we shall call an *operation of arity* $|I|$. We say that $\mathscr{L}^1(\mu)$ *is closed under* τ if τ returns functions in $\mathscr{L}^1(\mu)$ when applied to functions in $\mathscr{L}^1(\mu)$; i.e., for every $(f_i)_{i \in I} \subseteq \mathscr{L}^1(\mu)$, the function $\tau((f_i)_{i\in I})\colon \Omega \to \mathbb{R}$ given by $x \in \Omega \mapsto \tau((f_i(x))_{i\in I})$ belongs to $\mathscr{L}^1(\mu)$. If $\mathscr{L}^1(\mu)$ is closed under τ, we also say that τ *preserves integrability over* $(\Omega, \mathscr{F}, \mu)$. Finally, we say that τ *preserves integrability* if τ preserves integrability over every measure space.

Our first contribution is a solution to the following question, mentioned at the beginning.

Question 1. *Under which operations $\mathbb{R}^I \to \mathbb{R}$ are $\mathscr{L}^1$ spaces closed? Equivalently, which operations preserve integrability?*

For every $n \in \mathbb{N}$, we prove that a function $\tau\colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability precisely when τ is Borel measurable and *sublinear*, meaning that there are positive real numbers $\lambda_0, \dots, \lambda_{n-1}$ such that, for every $x \in \mathbb{R}^n$,

$$|\tau(x)| \leq \sum_{i=0}^{n-1} \lambda_i |x_i|.$$

We also prove an analogous result for the general case of arbitrary arity (not only for finite n), settling Question 1.

Examples of such operations are the constant 0, the addition $+$, the binary maximum $\vee$ and minimum $\wedge$, and, for $\lambda \in \mathbb{R}$, the scalar multiplication $\lambda(\,\cdot\,)$ by λ. A further example is the operation of countably infinite arity $\curlyvee$ defined as

$$\curlyvee(y, x_0, x_1, \dots) \coloneqq \sup_{n \in \omega}\{\min\{y, x_n\}\}.$$

Yet another example is the unary operation

$$\begin{aligned} \overline{\,\cdot\,}\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \overline{x} \coloneqq x \wedge 1, \end{aligned}$$

called *truncation*. Here, although the constant function 1 may fail to belong to $\mathscr{L}^1(\mu)$, it is always the case that $f \in \mathscr{L}^1(\mu)$ implies $\overline{f} \in \mathscr{L}^1(\mu)$.

Non-examples are the binary product and any non-zero constant.

We prove that the examples above are essentially "all" the operations that preserve integrability, in the sense that every operation that preserves integrability may be obtained from these by composition.

Moreover, we address a variation of Question 1 in which we restrict attention to finite measures.

In particular, for every $n \in \mathbb{N}$, a function $\tau\colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability over finite measure spaces precisely when it is Borel measurable and *subaffine*, meaning that there are positive real numbers $\lambda_0, \dots, \lambda_{n-1}, k$ such that, for every $x \in \mathbb{R}^n$,

$$|\tau(x)| \leq k + \sum_{i=0}^{n-1} \lambda_i |x_i|.$$

Furthermore, we prove that the operations that preserve integrability over finite measure spaces can be obtained by composition from 0, $+$, $\vee$, $\wedge$, $\lambda(\,\cdot\,)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and the constant 1 (which replaces the truncation operation $\overline{\,\cdot\,}$).

Truncated vector lattices and weak units

We investigate the equational laws satisfied by the operations that preserve integrability. We therefore work in the setting of varieties of algebras [4]. Under the term *variety* we include also infinitary varieties, i.e. varieties admitting primitive operations of infinite arity.

We assume familiarity with the basic theory of vector lattices, also known as vector lattices. All needed background can be found, for example, in the standard reference [8]. As usual, for a vector lattice G, we set $G^+ := \{x \in G \mid x \geq 0\}$.

A *truncated vector lattice* is a vector lattice G endowed with a function $\overline{\cdot}\colon G^+ \to G^+$, called *truncation*, which has the following properties for all $f, g \in G^+$.

(B1) $f \wedge \overline{g} \leq \overline{f} \leq f$.

(B2) If $\overline{f} = 0$, then $f = 0$.

(B3) If $nf = \overline{nf}$ for every $n \in \omega$, then $f = 0$.

The notion of truncation is due to R. N. Ball [3], who introduced it in the context of lattice-ordered groups.

Let us say that a partially ordered set B is *Dedekind σ-complete* if every nonempty countable subset $A \subseteq B$ that admits an upper bound admits a supremum. We prove that the category of Dedekind σ-complete truncated vector lattices is a variety generated by $\mathbb{R}$. This variety can be presented as having operations of finite arity together with the single operation $\curlyvee$ of countably infinite arity. Moreover, we prove that the variety is finitely axiomatisable by equations over the theory of vector lattices. One consequence is that the free Dedekind σ-complete truncated vector lattice over a set I (exists, and) is

$$\Big\{f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ is measurable and sublinear}\Big\}.$$

We prove analogous results for operations that preserve integrability over *finite* measure spaces. An element 1 of a vector lattice G is a *weak (order) unit* if $1 \geq 0$ and, for all $f \in G$, $f \wedge 1 = 0$ implies $f = 0$. We prove that the category of Dedekind σ-complete vector lattices with weak unit is a variety generated by $\mathbb{R}$, again with primitive operations of countable arity. It, too, is finitely axiomatisable by equations over the theory of vector lattices. We show that the free Dedekind σ-complete vector lattice with weak unit over a set I (exists, and) is

$$\Big\{f\colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ is measurable and subaffine}\Big\}.$$

The above presentations of the free algebras depend on a version of the Loomis-Sikorski Theorem for vector lattices, whose proof can be found in [7] (and can also be recovered from the combination of [5] and [6]). The theorem and its variants have a long history: for a fuller bibliographic account, please see [5].

This presentation is based on [2] (and partly on [1]).

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On Propositional Dynamic Logic and Concurrency

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Dynamic logics are families of logics where programs are part of the language of formulas itself, which enables the direct use of the logic to reason about the semantics of programs [11]. At the syntactical level, each program a defines the modalities $[a]$ and $\langle a \rangle$ and a formula $[a]\,\phi$ is interpreted as "every state reached after executing a satisfies the formula ϕ" while a formula $\langle a \rangle\,\phi$ is interpreted as "there is a state reached after executing a satisfying the formula F". Instances of dynamic logic include the modal μ-calculus [17], the Hennessy-Milner logic [12], and the propositional dynamic logic (PDL) [11], and provide solid foundations for the study of program verification and model checking [31, 7].

PDL and the concurrency problem

While PDL has been successfully applied to the study of sequential programs, extending this approach to concurrent programs has been proved to be challenging. In standard PDL, a program is represented by a regular expression that describes its set of possible traces. In other words, programs are elements of a free Kleene algebra. This representation of programs is satisfactory when reasoning about sequential programs, because one obtains that the theory of equational reasoning for Kleene algebras is a complete system for reasoning about *trace equivalence* [14, 18, 15, 30]. Trace equivalence is therefore captured by logical equivalence in PDL:

$$a \text{ and } b \text{ have the same traces} \quad \text{iff} \quad \vdash_{\mathsf{PDL}} [a]\,\phi \Leftrightarrow [b]\,\phi \text{ for any formula } \phi\,.$$

However, the case of concurrent programs with an interleaving semantics is more problematic. In the presence of interleaving, one expects traces differing by interleaving to be equivalent modulo equations of the form $a; b = b; a$ (called *commutations*). Unfortunately, the word problem in a Kleene algebra enriched with an equational theory containing such commutations is known to be undecidable[1] which makes undecidable checking whether two modalities in PDL are equivalent same.

As a consequence of this problem, applications of PDL to concurrency fall short of the expected level of expressivity from established theories, like CCS [23] and the π-calculus [24]. For example, previous works lack nested parallel composition, synchronisation, or recursion [21, 5, 28, 29, 27, 4]. In general, adding any new concurrency feature (e.g., a construct in the language of programs or a law defining its semantics) requires great care and effort

[1] In [16] is proven that the word problem in a star-continuous Kleene algebra can be reduced to an instance of the Post correspondence problem, by combining sequential composition, iteration, and commutations. This result has been recently extended to the general case of Kleene algebras [3].

in establishing the meta-theoretical properties of the logic. The result: a literature of various PDL, all independently useful, but with different limitations and dedicated technical developments.

In this talk

We discuss the result in [2], where we develop *operational propositional dynamic logic* (OPDL). The key innovation of OPDL is to distinguish and separate reasoning on programs from reasoning on their traces. Thanks to this distinction, we circumvent previous limitations and finally obtain a PDL that can be applied to established concurrency models, such as CCS [23] and choreographic programming [25]. Crucially, OPDL is a general framework: it is parameterised on the operational semantics used to generate traces from programs, yielding a simple yet reusable approach to characterise trace reasoning.

After recalling the axiomatization and semantics of PDL, we provide a proof of its soundness and completeness with respect to the non-wellfounded sequent calculus introduced in [8]. For this purpose, we provide the first cut-elimination result for this non-wellfounded calculus, by adapting the technique developed in [1].[2] This allows us to prove our results by reasoning on the axiomatisation and the sequent system, without directly relying on semantic arguments.

Then, we extend PDL with an additional axiom allowing us to encapsulate an operational semantics for a set of programs into the trace reasoning.

$$\mathsf{A}_{\mathscr{O}} \;:\; [\alpha]\,\phi \Leftrightarrow \left(\bigwedge_{\alpha \xrightarrow{b} \gamma} [b]\,[\gamma]\,\phi \right) \quad \text{with } \alpha \xrightarrow{b} \gamma \text{ in the operational semantics } \mathscr{O} \tag{0.1}$$

We call the resulting logic *operational propositional dynamic logic* (or OPDL), providing a general framework encompassing various previous works [21, 5, 10], and we provide instantiations of OPDL for Milner's CCS [23] and Montesi's latest presentation of choreographic programming [26].

We conclude by discussing the open questions about the axiomatizaiton of algebraic models for OPDL, and we provide a roadmap for future research in this area.

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C-completeness, u-presentability and Prucnal terms

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In this talk we explore a weakening of structural completeness that is obtained by considering specific subsets of the set of admissible quasiequations. We will also introduce certain terms, called *Prucnal terms* that are generalizations of the well-known concept of *ternary deductive term* introduced in [1]. We take for granted that the audience is aware of the general definition of structural completeness and primitivity and of the basic definitions and facts of universal algebra.

C-completeness

The concept of C-completeness has been introduced by the second author in [3]. Let A be any set; a **clone** of operations on A is a set of operations on A that contains all the projections and it is closed under composition (whenever possible). As the intersection of any family of clones is still a clone, it makes sense to talk about clone generation (that is of course a closure operator). If $\mathbf{A}$ is any algebra, then the **term clone** of $\mathbf{A}$, denoted by $\mathrm{Clo}(\mathbf{A})$, is the clone on $\mathbf{A}$ generated by all the fundamental operations.

Let now Q be a quasivariety; then the terms in the language of Q can be seen as operations on $\mathbf{F}_{\mathsf{Q}}(\omega)$, and the set of all terms is just the clone of all derived operations on $\mathbf{F}_{\mathsf{Q}}(\omega)$, i.e. the clone on $\mathbf{F}_{\mathsf{Q}}(\omega)$ generated by all the fundamental operations. We will refer to it as the **term clone** of Q and we will denote it by $\mathrm{Clo}(\mathsf{Q})$. Let C be a subclone of $\mathrm{Clo}(\mathsf{Q})$; a C**-quasiequation** is a quasiequation containing only operations from C. We say that Q is C**-structurally complete** if for every C-quasiequation Φ, if $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Phi$, then $\mathsf{Q} \vDash \Phi$. A quasivariety is C**-primitive** if all its subquasivarieties are C-structurally complete. Observe that if C' is a subclone of C and Q is C-structurally complete (C-primitive), then Q is C'-structurally complete (C'-primitive). Observe also that if T is a set of generators for C, it is easy to check that Q is C-structurally complete if and only if for every quasiequation Φ containing only operations from T, $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Phi$ entails $\mathsf{Q} \vDash \Phi$. Therefore if T is a set of terms that generates C we may talk about T-structural completeness and T-primitivity, meaning the corresponding concept for the clone generated by T. If C is the entire term clone of Q, then C-structural completeness is the usual structural completeness and C-primitivity is the usual primitivity.

Let Q be a quasivariety and let C be a subclone of the term clone of Q. Let Q^C be the class of all C-subreducts of algebras in Q; then it is easily seen that Q^C is a quasivariety in which all the C-quasiequation holding in Q are valid. So if Q^C is structurally complete or primitive, then Q is C-structurally complete or C-primitive. For instance if H is the variety of Heyting algebras, then its $\{\wedge, \to\}$-subreducts form the variety of Brouwerian semilattices, that is (as we have already observed) primitive; thus H is $\{\wedge, \to\}$-primitive.

The converse however fails to hold; the variety H of Heyting algebras is $\{\to, \neg\}$-structurally complete [6] but the quasivariety of its $\{\to, \neg\}$-subreducts is not structurally complete [2]. The problem is that there is a $\{\to, \neg\}$-quasiequation that is valid in the in $\mathbf{F}_{\mathsf{H}\{\to,\neg\}}(\omega)$ but it is not valid in $\mathbf{F}_{\mathsf{H}}(\omega)$.

u-presentability and Prucnal terms

Let $\mathbf{A}$ be any algebra, $\theta \in \mathrm{Con}(\mathbf{A})$ and let C be a subclone of $\mathrm{Clo}(\mathbf{A})$; by $\mathbf{A}^C$ we denote the algebra whose universe is A and whose fundamental operations are those in C. We say that θ is **u-presentable** relative to C if there is a set $\Delta \subseteq \mathrm{Con}(\mathbf{A})$ such that

1. $\theta = \bigcup \Delta$;
2. Δ is closed under finite joins;
3. $\mathbf{A}^C/\delta \in \mathbf{ISP}_u(\mathbf{A}^C)$ for all $\delta \in \Delta$;

and Δ is called a **u-presentation** of θ relative to C.

Theorem 1. *Let* $\mathbf{A}$ *be an algebra,* $\theta \in \mathrm{Con}(\mathbf{A})$ *and* C *a subclone of* $\mathrm{Clo}(\mathbf{A})$*; then the following are equivalent:*

1. *θ is u-presentable relative to C;*
2. $\mathbf{A}^C/\theta \in \mathbf{ISP}_u(\mathbf{A}^C)$.

Corollary 1. *For any algebra* $\mathbf{A}$ *and* $C \subseteq \mathrm{Clo}(\mathbf{A})$ *the following are equivalent:*

1. *every congruence of* $\mathbf{A}$ *is u-presentable relative to* C*;*
2. *every compact congruence of* $\mathbf{A}$ *is u-presentable relative to* C*.*

Now we can connect u-presentability and structural completeness.

Theorem 2. *Let* Q *be a quasivariety and* C *a clone of operations of* Q*; if every compact congruence of* $\mathbf{F}_{\mathsf{Q}}(\omega)$ *is u-presentable relative to* C*, then* Q *is* C*-structurally complete.*

Corollary 2. *Let* Q *be a quasivariety and* C *a clone of operations of* Q*; if every compact* Q*-congruence of every countably generated algebra in* Q *is u-presentable with respect to* C*, then* Q *is* C*-primitive.*

It is interesting to observe that if C is the entire term clone of Q, then we obtain a new necessary and sufficient condition for structural completeness.

Theorem 3. *A quasivariety* Q *is structurally complete if and only if every completely meet irreducible congruence* $\theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{F}_{\mathsf{Q}}(\omega))$ *is u-presentable.*

Corollary 3. *A quasivariety* Q *is primitive if and only if for every countably generated* $\mathbf{A} \in \mathsf{Q}$ *every completely meet irreducible* $\theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ *is u-presentable.*

Sometimes u-presentability is expressible directly via term operations. Let $\mathbf{A}$ be any algebra, C a subclone of the clone of all term operations on $\mathbf{A}$, T a set of generators for C and $\mathbf{A}^T$ the reduct of $\mathbf{A}$ to T; we say that $\mathbf{A}$ has the **Prucnal property** relative to C if for all $n \in \mathbb{N}$ there is a term $t_n(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ such that for any compact $\theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ such that $\theta = \bigvee_{i=1}^{n} \vartheta_{\mathbf{A}}^{\mathsf{Q}}(a_i, b_i)$

1. the map $\sigma_n : c \longmapsto t_n(a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ is an endomorphism of $\mathbf{A}^T$;
2. $\ker(\sigma_n) = \theta$;

the terms t_n are called the **Prucnal terms** relative to C and the endomorphisms σ_n are called the **Prucnal C-endomorphisms**. If C is the entire clone of derived operations on $\mathbf{A}$, then we will drop the decoration C.

Theorem 4. *Let* Q *be a quasivariety and let* C *a clone of term operations of* Q*. If* $\mathbf{F}_{\mathsf{Q}}(\omega)$ *has the Prucnal property relative to* C*, then* Q *is* C*-structurally complete.*

Corollary 4. *Let* Q *be a quasivariety and let* C *a clone of term operations of* Q*. If every countably generated algebra in* Q *has the Prucnal property relative to* C*, then* Q *is* C*-primitive.*

Principal Prucnal property

A quasivariety Q has the **principal Prucnal property** relative to C, if there is a term $t(x, y, z)$ that is a Prucnal term for principal congruences, relative to C, i.e. for all $\mathbf{A} \in \mathsf{Q}$ and for all $a, b, \in A$

1. the map $\sigma : c \longmapsto t(a, b, c)$ is an endomorphism of $\mathbf{A}^C$;
2. $\ker(\sigma) = \vartheta_{\mathbf{A}}^{\mathsf{Q}}(a, b)$.

We will show that for any quasivariety the principal Prucnal property (relative to C) implies the Prucnal property (relative to C). To this aim we need several lemmas.

Lemma 1. *[4] Let* Q *be a quasivariety and* $\mathbf{A} \in \mathsf{Q}$*; if* $\theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ *and* $a, b \in A$ *then* $(\theta \vee \vartheta_{\mathbf{A}}^{\mathsf{Q}}(a, b))/\theta = \vartheta_{\mathbf{A}/\theta}(a/\theta, b/\theta)$.

For the following lemma, in order to avoid clutter we use a special notation; first we will denote a sequence $a_1, \ldots, a_n \in A$ by $\mathbf{a}^n$. Next if $\theta \in \mathrm{Con}(\mathbf{A})$ we will write $\bar{\mathbf{A}}$ for $\mathbf{A}/\theta$, $\overline{a}$ for a/θ and $\overline{\mathbf{a}}^n$ for $\overline{a}_1, \ldots, \overline{a}_n$.

Lemma 2. *Let* Q *be a quasivariety,* $\mathbf{A} \in \mathsf{Q}$ *and* $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$*. If* $\overline{x} = x/\vartheta_{\mathbf{A}}^{\mathsf{Q}}(a_n, b_n)$*,* $\bar{\mathbf{A}} = \mathbf{A}/\vartheta_{\mathbf{A}}^{\mathsf{Q}}(a_n, b_n)$ *and* $c, d \in A$ *then*

$$(c, d) \in \vartheta_{\mathbf{A}}^{\mathsf{Q}}(a_1, b_1) \vee \ldots \vee \vartheta_{\mathbf{A}}^{\mathsf{Q}}(a_n, b_n) \quad \textit{if and only if}$$
$$(\overline{c}, \overline{d}) \in \vartheta_{\bar{\mathbf{A}}}^{\mathsf{Q}}(\overline{a}_1, \overline{b}_1) \vee \ldots \vee \vartheta_{\bar{\mathbf{A}}}^{\mathsf{Q}}(\overline{a}_{n-1}, \overline{b}_{n-1}).$$

Let Q be a quasivariety with a principal Prucnal term relative to C, say $t(x, y, z)$. We define for $n \geq 1$

$$t_1(x_1, y_1, z) := t(x_1, y_1, z)$$
$$t_{n+1} = t_n(x_1, \ldots, x_n, y_1, \ldots, y_n, t(x_n, y_n, z)).$$

Lemma 3. *Let Q be a quasivariety with principal Prucnal term $t(x, y, z)$ relative to C, $\mathbf{A} \in \mathsf{Q}$ and $a, b, c, d \in A$. Then*

$$(c, d) \in \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_1, b_1) \vee \ldots \vee \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_n, b_n) \quad \textit{if and only if}$$
$$t_n(\mathbf{a}^n, \mathbf{b}^n, c) = t_n(\mathbf{a}^n, \mathbf{b}^n, d).$$

We observe that Lemma 3 is a generalization of Theorem 2.6 in [1].

Corollary 5. *If a quasivariety Q has a principal Prucnal term relative to C, then it has the Prucnal property relative to C.*

Proof. The terms $t_n, n \geq 1$ clearly satisfy the first condition for Prucnal terms, as iterated compositions of t. By Lemma 3 they also satisfy the second. □

Ternary deduction terms

Here we will consider a special principal Prucnal term. Let Q be a quasivariety; a **ternary deductive term (TD-term)** for Q is a ternary term $t(x, y, z)$ such that

- $\mathsf{Q} \vDash t(x, x, z)$;
- if $(c, d) \in \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a, b)$ then $t(a, b, c) = t(a, b, d)$.

Let Q be a quasivariety with a TD-term $t(x, y, z)$ and q a k-term of Q; we say that q **commutes with** t if for all $\mathbf{A} \in \mathsf{Q}$ and $a, b, c_1, \ldots, c_k \in A$

$$t(a, b, q(c_1, \ldots, c_k)) = q(t(a, b, c_1), \ldots, t(a, b, c_k)).$$

Theorem 5. *A quasivariety haveing a TD-term is a variety.*

Theorem 6. *If $t(x, y, z)$ is a TD-term for Q, then t is a principal Prucnal term relative to any clone C of operations that commute with $t(x, y, z)$.*

Corollary 6. *Let Q be a variety with a TD-term. Then for all nontrivial $\mathbf{A} \in \mathsf{Q}$, $\mathbf{A}$ has the Prucnal property for $\mathbf{A}$, relative to any clone C of operations that commute with the relative TD-term.*

Corollary 7. *Let Q be a quasivariety with a relative TD-term; then Q is C-primitive for any clone C of terms that commute with the relative TD-term.*

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Esakia Duality for Temporal Heyting Algebras

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The temporal Heyting calculus **tHC**, first presented in [3], is the natural temporal augmentation of the modalized Heyting calculus **mHC**, also first studied in [3]. It has as its algebraic models the category of temporal Heyting algebras **tHA**, a class of Heyting algebras with a "forward-looking" $\Box$ that has a left-adjoint, "backward-looking" $\blacklozenge$. Being intuitionstic, however, it lacks the modalities $\Diamond$ and $\blacksquare$ typically defined in terms of negation.

	Past	Future
$\exists$	$\blacklozenge$	$\Diamond$
$\forall$	$\blacksquare$	$\Box$

We present a suitable category of topological models for the logic (temporal Esakia spaces **tES**) and develop an Esakia duality between the categories of algebraic and topological models. This includes defining a class of filters on **tHA** and closed upsets on **tES** such that we have a poset-isomorphism between **tHA** congruences, our class of filters, and our class of closed upsets. Having achieved this, we classify simple and subdirectly-irreducible (s.i.) temporal Heyting algebras via their dual spaces as was done for "BAOs" in [4] and "distributive modal algebras" in [1]. Finally, we use this characterization to achieve a relational completeness result combining finiteness (achieved via finite model property) and a notion of "rootedness" dual to subdirect-irreducibility (analogous to the result that **IPC** is complete with respect to the class of finite trees).

Logic

Definition 1. The *modalized Heyting calculus* **mHC** is the smallest extension of **IPC** containing the following axioms and closed under modus ponens.

$$\Box(p \to q) \to (\Box p \to \Box q) \qquad p \to \Box p \qquad \Box p \to (q \vee q \to p)$$

The *temporal Heyting calculus* **tHC** is the smallest extension of **mHC** containing the following axioms and closed under modus ponens and the rule $\frac{\varphi \to \chi}{\blacklozenge\varphi \to \blacklozenge\chi}$.

$$\blacklozenge(p \vee q) \to (\blacklozenge p \vee \blacklozenge q) \qquad \blacklozenge\bot \to \bot \qquad p \to \Box\blacklozenge p \qquad \blacklozenge\Box p \to p$$

Algebraic models

We define the algebraic models for our logic **tHC**.

Definition 2. A *temporal Heyting algebra* is a frontal Heyting algebra $\mathbb{A}$ (see [2]) with an additional operator $\blacklozenge : \mathbb{A} \to \mathbb{A}$ such that $\blacklozenge \dashv \Box$, ie.

$$\blacklozenge a \leqslant b \quad \Longleftrightarrow \quad a \leqslant \Box b.$$

The category **tHA** has as its objects temporal Heyting algebras and as its morphisms algebraic homomophisms.

Here we define the class of filters that will correspond to congruences on our algebras $\mathsf{Cong}(\mathbb{A})$, analogous to the correspondence between congruences and "open filters" on BAOs [5, Theorem 29]. We also define a class of elements analogous to "open elements".

Definition 3. Given $\mathbb{A} \in \mathbf{tHA}$, a $\blacklozenge$*-filter* is a filter $F \subseteq \mathbb{A}$ such that

$$a \to b \in F \quad \Longrightarrow \quad \blacklozenge a \to \blacklozenge b \in F.$$

A $\blacklozenge$*-compatible element* is an element $a \in \mathbb{A}$ such that for all b,

$$a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b).$$

The sets of $\blacklozenge$-filters and $\blacklozenge$-compatible elements of $\mathbb{A}$ are denoted by $\blacklozenge\mathsf{Filt}(\mathbb{A})$ and $\blacklozenge\mathsf{Com}(\mathbb{A})$ respectively.

Theorem 1. *Given* $\mathbb{A} \in \mathbf{tHA}$,

$$\langle \mathit{Cong}(\mathbb{A}), \subseteq \rangle \cong^{\mathbf{POS}} \langle \blacklozenge \mathit{Filt}(\mathbb{A}), \subseteq \rangle.$$

In the finite case, the correspondence can be given element-wise.

Corollary 1. *Given* $\mathbb{A} \in \mathbf{tHA}_{\mathit{fin}}$,

$$\langle \mathit{Cong}(\mathbb{A}), \subseteq \rangle \cong^{\mathbf{POS}} \langle \blacklozenge \mathit{Filt}(\mathbb{A}), \subseteq \rangle \cong^{\mathbf{POS}} \langle \blacklozenge \mathit{Com}(\mathbb{A}), \geqslant \rangle.$$

Topological models

We define the topological models for our logic **tHC**.

Definition 4. A *temporal Esakia space* is an "Rf-Heyting space" $\mathbb{X}$ (see [2]) with an additional "backward-looking" relation $R^{\triangleleft} \subseteq \mathbb{X} \times \mathbb{X}$ such that

- $R^{\triangleleft}$ is inverse to $R^{\triangleright}$ (where $R^{\triangleright}$ is the "forward-looking" relation)
- $K \in \mathsf{ClopUp}(\mathbb{X})$ implies $R^{\triangleright}[K] \in \mathsf{ClopUp}(K)$
- $R^{\triangleright}[x]$ is closed.

A *temporal Esakia morphism* is an "Rf-Heyting morphism" $f : \mathbb{X} \to \mathbb{Y}$ (see [2]) such that

$$fx_2 \; R^{\triangleleft} \; y \text{ implies } \exists x_1 \in \mathbb{X} \text{ such that } x_2 \; R^{\triangleleft} \; x_1 \text{ and } y \leqslant fx_1.$$

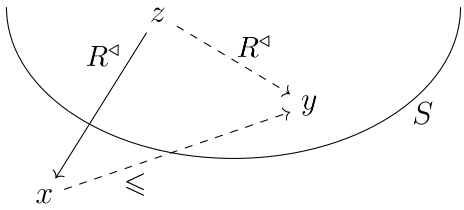
The category **tES** has as its objects temporal Esakia spaces and as its morphisms temporal Esakia morphisms.

Here we define a class of subsets on temporal Esakia spaces that will correspond to congruences of the dual algebras.

Definition 5. Given $\mathbb{X} \in \mathbf{tES}$, we call a subset $S \subseteq \mathbb{X}$ *archival* if

$$x \notin S \ni z \text{ and } z\, R^{\triangleleft}\, x \quad \Longrightarrow \quad R^{\triangleleft}[z] \cap {\uparrow}x \cap S \neq \varnothing.$$

This is depicted as follows.



We denote the set of archival subsets of $\mathbb{X}$ by $\mathsf{Arc}(\mathbb{X})$, the set of archival *upsets* of $\mathbb{X}$ by $\mathsf{ArcUp}(\mathbb{X})$, and the set of *closed* archival upsets of $\mathbb{X}$ by $\mathsf{ClArcUp}(\mathbb{X})$.

We define a notion of reachability on our spaces in terms of closed archival upsets. This is essentially analogous to the "specialization order".

Definition 6. Given $\mathbb{X} \in \mathbf{tES}$,

$$x \trianglelefteq y \quad :\Longleftrightarrow \quad y \in \bigcap\{C \in \mathsf{ClArcUp}(\mathbb{X}) \mid x \in C\}.$$

If $x \trianglelefteq y$, we say that y *is topo-reachable by* x. We denote the set of topo-roots of $\mathbb{X}$ (ie. the points that are roots with respect to $\trianglelefteq$) by $\mathsf{ToRo}(\mathbb{X})$.

In the finite case, we can define another notion of reachability (via a relation Z) only in terms of the underlying frame.

Definition 7. Given $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we define the following relation B.

$$x\, B\, w \quad :\Longleftrightarrow \quad w \leqslant x \text{ and } (w, x] \cap \mathsf{Refl}(\mathbb{X}) = \varnothing$$

We also define the following relations for all $n \in \mathbb{N}$.

$$Z_0 := \Delta_{\mathbb{X}} \qquad Z_{n+1} := Z_n; B; \leqslant \qquad Z := \bigcup_{m \in \mathbb{N}} Z_m$$

In the finite case, we show that these two notions of reachability are equivalent.

Proposition 1. *Given $\mathbb{X} \in \mathbf{tES}_{\mathit{fin}}$, $x \trianglelefteq y$ iff $x\, Z\, y$. Note that this implies that x is a topo-root iff x is a Z-root as well as the fact that $\mathbb{X}$ is topo-connected iff $\mathbb{X}$ is Z-connected.*

Esakia Duality

Building on the work in [2], we further augment the functors $◌_* : \mathbf{HA} \leftrightarrows \mathbf{ES} : ◌^*$ using $\blacklozenge$ and $R^{\triangleleft}$ to define eachother in the standard way. We then prove a duality between the categories of our algebraic and topological models.

Theorem 2. $\mathbf{tHA} \cong \mathbf{tES}^{op}$.

We then extend our congruence-correspondence to the dual spaces.

Theorem 3. *Given $\mathbb{A} \in \mathbf{tHA}$,*

$$\langle \mathit{Cong}(\mathbb{A}), \subseteq\rangle \cong^{\text{POS}} \langle \blacklozenge \mathit{Filt}(\mathbb{A}), \subseteq\rangle \cong^{\text{POS}} \langle \mathit{ClArcUp}(\mathbb{A}_*), \supseteq\rangle.$$

Characterizations

We characterize *simple* algebras lattice-theoretically and order-topologically.

Theorem 4. *Given $\mathbb{A} \in \mathbf{tHA}$, the following are equivalent.*

- $\mathbb{A}$ *is simple*
- $\blacklozenge \mathsf{Filt}(\mathbb{A}) = \{\{1\}, \mathbb{A}\}$
- $\mathbb{A}_*$ *is topo-connected*

In the finite case, we can give the same characterization element-wise and frame-theoretically.

Corollary 2. *Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent.*

- $\mathbb{A}$ *is simple*
- $\blacklozenge \mathsf{Com}(\mathbb{A}) = \{1, 0\}$
- $\mathbb{A}_*$ *is Z-connected*

We characterize *subdirectly-irreducible* algebras lattice-theoretically and order-topologically.

Theorem 5. *Given $\mathbb{A} \in \mathbf{tHA}$, the following are equivalent.*

- $\mathbb{A}$ *is s.i.*
- $\blacklozenge \mathsf{Filt}(\mathbb{A})$ *has a second-least element*
- $\mathsf{ToRo}(\mathbb{A}_*)$ *non-empty and open*

In the finite case, we can again give the same characterization element-wise and frame-theoretically.

Corollary 3. *Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent.*

- $\mathbb{A}$ *is s.i.*
- $\blacklozenge \mathsf{Com}(\mathbb{A})$ *has a second-greatest element*
- $\mathbb{A}_*$ *is Z-rooted*

Applications to tHC

We show that **tHC** has the finite model property, implying the following, stronger algebraic completeness result.

Theorem 6. *The logic **tHC** is sound and complete with respect to the class of finite, subdirectly-irreducible temporal Heyting algebras $\mathbf{tHA}_{\mathsf{fsi}}$.*

Finally, we use our characterization of subdirectly-irreducible temporal Heyting algebras to arrive at the following relational completeness result.

Theorem 7. *The logic **tHC** is sound and complete with respect to the class of finite, Z-rooted "temporal transits" (transits [3] with an inverse relation $R^{\triangleleft}$).*

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Proof Compression via Subatomic Logic and Guarded Substitutions

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The affinity between structural proof theory and the mathematical foundations of computation establishes mechanisms of proof compression as a natural object of study. The most prominent proof compression mechanism is the *cut rule* [11, 12], which allows lemmas to be reused in a proof. And indeed, eliminating the cuts from a proof can lead to a non-elementary blow-up [6]. In propositional logic, this blow-up is still exponential [19].

In the area of proof complexity, a distinct subfield of proof theory, other mechanisms of proof compression have been studied, the most notable ones being substitution [9] and Tseitin extension [20]. In both cases, proof compression is achieved by permitting propositional variables to replace arbitrary subformulae in a proof. The cost of eliminating either rule from a proof is an exponential blow-up (and it is not known whether this can be done more efficiently).

Given their conceptual similarity, it is perhaps not so surprising that, in the presence of cut, extension and substitution are *p-equivalent*, i.e., a system with cut and substitution can polynomially simulate one with cut and Tseitin extension [9], and vice versa [16]. This has been shown in the setting of Frege systems, which always contain the cut because of the presence of the *modus ponens* rule. Moreover, even in the absence of cut, it has also been demonstrated that the two proof compression mechanisms of substitution and Tseitin extension are p-equivalent [18, 17]. However, it is not known if cut-free systems with extension or substitution can p-simulate systems with cut and extension or substitution.

This leaves us with two powerful proof compression mechanisms: (i) the cut and (ii) extension/substitution. It is an open question whether one of them subsumes the other, or whether they are truly independent.

In this work we give a surprising answer to this question. We observe that both proof compression mechanisms are subsumed by a more general one, namely, *guarded substitution*, which is a variant of explicit substitutions [1].

To see how this is possible, let us first observe that the proof compression of cut and substitution comes from the ability of reusing information. And the most basic inference rules that deal with duplication of information are the rules of *contraction* and *cocontraction* shown below:[1]

[1]In fact, the combination of contraction and cocontraction form another mechanism of proof compression, when cut is absent. This has been investigated in [15, 8, 10]. However, we will not go into further details of this, as we develop in this work a cut-free system that can p-simulate cut.

$$\mathsf{cut}\frac{\vdash \Gamma, A \qquad \vdash \bar{A}, \Delta}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{i}\!\uparrow\frac{A \wedge \bar{A}}{0} \quad \rightsquigarrow \quad \mathsf{ai}\!\uparrow\frac{a \wedge \bar{a}}{0} \quad \rightsquigarrow \quad \hat{\mathsf{a}}\wedge\frac{(0\,\boldsymbol{a}\,1) \wedge (1\,\boldsymbol{a}\,0)}{(0 \wedge 1)\,\boldsymbol{a}\,(1 \wedge 0)}$$

Figure 1: Evolution of cut from the sequent calculus via deep inference to subatomic proof theory

$$\mathsf{c}\!\downarrow\frac{A \vee A}{A} \qquad \text{and} \qquad \mathsf{c}\!\uparrow\frac{A}{A \wedge A} \tag{0.1}$$

A move to a deep inference proof system [14, 7], which allow for finer granularity in the design of the inference rules, enables the restriction of these rules to their atomic form, shown below, provided there is an additional purely linear inference rule in the system [7]. This rule, called *medial*, is shown in the middle below:

$$\mathsf{ac}\!\downarrow\frac{a \vee a}{a} \qquad \mathsf{m}\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{ac}\!\uparrow\frac{a}{a \wedge a} \tag{0.2}$$

This leads to the proof system SKS [7], consisting only of atomic rules (like ac↓ and ac↑ above) that change the size of a formula, and purely linear rules (like m above) that only rearrange subformulae without changing the size.

The next insight comes from the concept of *subatomic proof theory* [2, 3, 5] which splits the atoms into binary connectives. A formula "$A\,\boldsymbol{a}\,B$" is then interpreted as "if a is false then A, and if a is true then B". In this setting, we can write a as $0\,\boldsymbol{a}\,1$ and its dual $\bar{a}$ as $1\,\boldsymbol{a}\,0$. The two rules of ac↓ and ac↑ from above now become:

$$\frac{(0\,\boldsymbol{a}\,1) \vee (0\,\boldsymbol{a}\,1)}{(0 \vee 0)\,\boldsymbol{a}\,(1 \vee 1)} \qquad \text{and} \qquad \frac{(0 \wedge 0)\,\boldsymbol{a}\,(1 \wedge 1)}{(0\,\boldsymbol{a}\,1) \wedge (0\,\boldsymbol{a}\,1)} \tag{0.3}$$

which are just instances of the general rules

$$\check{\vee}\boldsymbol{a}\frac{(A\,\boldsymbol{a}\,B) \vee (C\,\boldsymbol{a}\,D)}{(A \vee C)\,\boldsymbol{a}\,(B \vee D)} \qquad \text{and} \qquad \hat{\wedge}\boldsymbol{a}\frac{(A \wedge B)\,\boldsymbol{a}\,(C \wedge D)}{(A\,\boldsymbol{a}\,C) \wedge (B\,\boldsymbol{a}\,D)} \tag{0.4}$$

which have the same shape as the medial rule in (0.2) above. The same principle also applies to the cut rule: in Figure 1 we see the evolution of the cut, starting from the sequent calculus, first becoming an atomic rule, and finally a subatomic rule. In this way we can obtain a proof system for propositional logic in which *all* inference rules are linear rewriting steps [2], except for the rules dealing with the units, for example

$$=\frac{A}{A \wedge 1} \qquad =\frac{A}{A \vee 0} \qquad =\frac{A \wedge 1}{A} \qquad =\frac{A \vee 0}{A} \tag{0.5}$$

Even though these are rather trivial inference steps in a standard proof system, they are the only ones that break a rigidly defined notion of linearity in a subatomic proof system. To achieve what is called a strictly linear system, these can be eliminated, but the naive way of doing so leads to an exponential blow-up of the size of the proof [4, 5]. However, by allowing explicit substitutions as constructors in formulae and derivations, the size of the proof expansion can be reduced to a polynomial [5]. The resulting system (called

KDTS in [5]) is p-equivalent to SKS (and therefore also to standard Frege systems without extension or substitution) and still contains the cut (in its linear form, as shown on the right in Figure 1). And, unsurprisingly, eliminating the cut from this system leads to an exponential blow-up [5]. Furthermore, it is unknown whether these explicit substitutions can in any way polynomially simulate Tseitin extension or substitution in Frege systems.

In other words, in terms of proof complexity, nothing has been gained so far with respect to what we said at the beginning of this introduction; even in the unfamiliar climes of a subatomic proof system with explicit substitutions, it appears that both cut and extension/substitution operate as independent means of compressing proofs.

This motivates our paper, in which we introduce *guarded substitutions* that offer us a distinct new means of proof compression.

Guarded substitutions are a variant of explicit substitution that, instead of representing the replacement of every occurrence of a free variable, only select a certain subset of the free occurrences of the given variable. To make this formal, we assign to every variable occurrence a *range*, and to every guarded substitution a *guard*, and the substitution can apply to the variable occurrence, if the guard is in the range. For example, the formula $\langle A|x\rangle((x \wedge x) \vee (y \wedge x))$ with an ordinary explicit substitution becomes $(A \wedge A) \vee (y \wedge A)$ when the substitution is carried out, whereas the formula $\langle\!\langle A|x\rangle\!\rangle^p((x^{q,p} \wedge x^r) \vee (y^p \wedge x^{p,r}))$ becomes $(A \wedge x^r) \vee (y^p \wedge A)$ when the substitution is carried out, because only the first and the last x have the guard p in its range.

With this additional construct, the new system, that we call KSubG, can polynomially simulate Frege systems with substitution. Furthermore, we can do this with the cut-free fragment KSubG⁻. In other words, guarded substitution can polynomially simulate the cut, as well as Tseitin extension and substitution in Frege systems. And surprisingly, this can even be done when we only allow the units 0 and 1 in the place of the formula to be substituted (the A in the example above).

Since Frege systems with substitution are known to be the most potent propositional proof systems, in the sense that they can polynomially simulate every other proof system for classical propositional logic, our system KSubG has now the same property, with the additional feature that every inference step is linear. There is never an inference that adds or deletes information.

This becomes possible because a subatomic derivation can be interpreted as a *superposition* of standard derivations. Even though this idea has been around since the beginning of subatomic proof theory [13], only our guarded substitutions allow to make use of this. We can see such a superposition as executing several similar shaped derivations in parallel, and the guarded substitutions can be used to read out the correct results.

For example, the derivation

$$\Psi = (x \vee x) \wedge \boxed{\mathsf{mix}\,\frac{y \wedge 0}{y \vee 0}}$$

is a superposition of the derivation

$$\Phi = \mathsf{ai}{\uparrow}\,\frac{\boxed{\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}} \wedge \boxed{\mathsf{aw}{\downarrow}\,\frac{0}{\bar{a}}}}{0}$$

because by substituting 0 for x and 1 for y in Ψ we can recover the result of assuming that a is false in Φ, and by substituting 1 for x and 0 for y we can recover the result of assuming that a is true. Then Φ can be encoded by the subatomic derivation with guarded substitutions

$$\langle\!\langle 0|x, 1|y \rangle\!\rangle^l \, \langle\!\langle 1|x, 0|y \rangle\!\rangle^r \, \langle \Psi|z \rangle (z^l \, \boldsymbol{a} \, z^r)$$

We can recover from this the conclusion of Φ, and the resulting derivation contains no cuts.

This allows for the factorisation of lemmas with different inputs, essentially performing the work of both modus ponens and the Frege substitution rule. In this way, we are able to p-simulate substitution Frege in a cut-free subatomic system with guarded substitutions.

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The modal logic of classes of structures

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In [1] Hamkins and Löwe investigate how the set theoretical method of forcing between models of set theory affects the corresponding theory of a model. The method of forcing, which has become a fundamental tool in set theory, was first introduced by Paul Cohen in 1962 in order to prove the indepence of the Axiom of Choice and the Continuum Hypothesis from the other axioms of set theory. Since then, this method has widely been used to construct a huge variety of models of set theory and prove many other independence results.

With forcing one builds an extension of any model of set theory using algebraic tools; the resulting forcing extension will be another model which is closely related to the original one, but may exhibit different set theoretical truths in a way that can often be carefully controlled. Since the ground model has some access, via the forcing relation, to the truths of the forcing extension, there are clear affinities between forcing and modal logic. In fact, one can even consider the collection of all models of set theory, where the accessibility relation is induced by forcing, as an enormous Kripke model. Following this strategy, they define that a statement of set theory φ is possible if it holds in some forcing extension and necessary if it holds in all forcing extensions; the modal notations $\Diamond\varphi$ and $\Box\varphi$ express respectively that φ is possible and necessary.

More specifically, a **modal assertion** is a formula of propositional modal logic, which is expressed with propositional variables p_i, the usual Boolean connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\iff$ and the modal operators $\Box$, $\Diamond$. The notation $\varphi(p_0, ..., p_n)$ is used to denote a modal assertion whose propositional variables are among p_0,...,p_n. A modal assertion $\varphi(p_0, ..., p_n)$ is a **valid principle of forcing** if for all sentences ψ_i in the language of set theory, $\varphi(\psi_0, ..., \psi_n)$ holds under the forcing interpretation of $\Box$ and $\Diamond$. We say that $\varphi(p_0, ..., p_n)$ is a **ZFC provable principle of forcing** if ZFC proves all the substitution instances $\varphi(\psi_0, ..., \psi_n)$. In their paper, Hamkins and Löwe prove that if ZFC is consistent, then the ZFC-provably valid principles of forcing are exactly the assertions of the well-known modal logic S4.2: this is what the authors of [1] mean when they assert that the modal logic of forcing is S4.2.

A natural extension of the problem introduced by Hamkins and Löwe was presented in [2]: the key idea of this paper is to consider a class of structures $\mathfrak{C}$ endowed with a binary relation $\sqsubseteq$ which is interpreted as accessibility: given $\mathbf{M}$, $\mathbf{N} \in \mathfrak{C}$, the notation $\mathbf{M} \sqsubseteq \mathbf{N}$ is used to state that $\mathbf{M}$ accesses $\mathbf{N}$. Clearly, this gives $(\mathfrak{C}, \sqsubseteq)$ the structure of a Kripke frame, whose Kripke models we can study. It is then natural to study the modal logic this interpretation gives rise to.

First of all, we consider the case in which $\mathfrak{C}$ is generic. Let $\mathfrak{C}$ be any class and $\sqsubseteq$ be a definable binary class relation on $\mathfrak{C}$. We consider $(\mathfrak{C}, \sqsubseteq)$ as a Kripke frame; a valuation is a function $v : \mathit{Prop} \times \mathfrak{C} \rightarrow \{0, 1\}$ (where we denote by Prop the set of propositional variables) and a Kripke model is a triple $(\mathfrak{C}, \sqsubseteq, v)$. The **Kripke semantics** for the language $\mathscr{L}_\Box$ of modal logic can be easily defined. If $\mathbf{M} \in \mathfrak{C}$, then:

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models p$ if and only if $v(p, \mathbf{M}) = 1$ (if p is a propositional variable);

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \varphi \wedge \psi$ if and only if $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \varphi$ and $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \psi$;

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \neg\varphi$ if and only if $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \not\models \varphi$;

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \Box\varphi$ if and only if for all $\mathbf{N} \in \mathfrak{C}$ such that $\mathbf{M} \sqsubseteq \mathbf{N}$ we have $\mathfrak{C}, \sqsubseteq, v, \mathbf{N} \models \varphi$.

A modal formula φ is **valid in a Kripke model** $(\mathfrak{C}, \sqsubseteq, v)$ if for every $\mathbf{M} \in \mathfrak{C}$ we have $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \varphi$. A modal formula φ is **valid in a Kripke frame** $(\mathfrak{C}, \sqsubseteq)$ if it is valid in every model based on that frame. We call the **modal logic of** $(\mathfrak{C}, \sqsubseteq)$, denoted by $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$, the collection of modal formulas which are valid in $(\mathfrak{C}, \sqsubseteq)$.

The problem proposed in [2] concerns characterizing for a given frame $(\mathfrak{C}, \sqsubseteq)$ the modal logic $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ in terms of other well-known modal logics, basing on the study of the class $\mathfrak{C}$ and of the properties of the relation $\sqsubseteq$. This problem becomes more interesting when we consider $\mathfrak{C}$ as a class of structures and investigate modal logic it gives rise to. Therefore, we go on providing the general setting for classes of structures.

Let S be a non-logical vocabulary, $\mathscr{L}_S$ be the first order language with vocabulary S and let $\mathfrak{C}$ be a class of $\mathscr{L}_S$-structures (for example $\mathfrak{C}$ could be the class of all $\mathscr{L}_S$-structures satisfying a collection of $\mathscr{L}_S$-sentences). A language $\mathscr{L} \supseteq \mathscr{L}_S$ is called **$\mathfrak{C}$-adequate** if there is a definable model relation $\models$ between the elements of $\mathfrak{C}$ and $\mathscr{L}$ sentences, which extends the usual model relation od $\mathscr{L}_S$.

An **$\mathscr{L}$-translation** is a function $T : Prop \to Sent(\mathscr{L})$, assigning an $\mathscr{L}$-sentence to each propositional variable. Any $\mathscr{L}$-translation gives rise to a valuation v_T for the class $\mathfrak{C}$, called the **$\mathscr{L}$-structure valuation** in a natural way: $v_T(p, \mathbf{M}) = 1$ if and only if $\mathbf{M} \models T(p)$. Clearly, this induces a Kripke model $(\mathfrak{C}, \sqsubseteq, v_T)$.

The **$\mathscr{L}$-structure modal logic** of $(\mathfrak{C}, \sqsubseteq)$, denoted by $\mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq)$ is the set of modal formulas that are valid in each Kripke model $(\mathfrak{C}, \sqsubseteq, v_T)$ for an $\mathscr{L}$-translation T. Notice that

$$\mathsf{ML}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{ML}_{\mathscr{L}_S}(\mathfrak{C}, \sqsubseteq).$$

Now let $\mathfrak{C}$ and $\sqsubseteq$ be respectively a fixed class of structures and a binary relation on $\mathfrak{C}$. The problem of showing that $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ is some well-known modal logic L can be easily split in two tasks: proving that L is a lower bound for $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ and then showing that the lower bound is also an upper bound.

Finding a lower bound is quite easy: the strategy is based on the classical results concerning completeness of some modal logics with respect to certain classes of frames. Consider for example $\mathsf{S4.2}$, that is known to be complete with respect to the class of frames in which the accessibility relation is reflexive, transitive and directed: if we manage to prove that the relation $\sqsubseteq$ on $\mathfrak{C}$ is a directed pre-order, then we obtain $\mathsf{S4.2} \subseteq \mathsf{ML}(\mathfrak{C}, \sqsubseteq)$.

The same argument can be applied to other well-known modal logics, depending on the properties of the relation $\sqsubseteq$. In particular, if we want our logic $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ to be at least $\mathsf{S4}$, we need to request that $\sqsubseteq$ is reflexive and transitive on $\mathfrak{C}$. In other words, this happens if the operator which sends each $\mathbf{M} \in \mathfrak{C}$ to $\{\mathbf{N} \in \mathfrak{C} : \mathbf{M} \sqsubseteq \mathbf{N}\}$ gives rise to a closure operator.

The task of finding upper bounds and in particular proving that the lower bound is also an upper bound requires more effort and is based on finding a labeling for a fixed frame using $(\mathfrak{C}, \sqsubseteq)$. More specifically, if $(\mathscr{F}, \mathscr{R})$ is a transitive and reflexive frame with initial world w_0 (i.e. $w_0 \mathscr{R} u$ for every $u \in \mathscr{F}$), then a **$\mathfrak{C}$-labeling** of the rooted frame $(\mathscr{F}, \mathscr{R}, w_0)$ for an element $\mathbf{M} \in \mathfrak{C}$ is an assignment to each node $w \in \mathscr{F}$ of a formula φ_w in the language $\mathscr{L}$, such that:

1. every $\mathbf{N} \in \mathfrak{C}$ such that $\mathbf{M} \sqsubseteq \mathbf{N}$ satisfies exactly one φ_w;
2. if $\mathbf{N} \in \mathfrak{C}$ is such that $\mathbf{M} \sqsubseteq \mathbf{N}$ and $\mathbf{N} \models \varphi_w$, then $\mathbf{N} \models \Diamond \varphi_u$ if and only if $w \mathscr{R} u$;
3. $\mathbf{M} \models \varphi_{w_0}$.

We can show (a proof can be found in [1]) that if for a fixed frame $(\mathscr{F}, \mathscr{R})$ satisfying the hypothesis above and for a given initial world $w_0 \in \mathscr{F}$ there exists a $\mathfrak{C}$-labeling for every $\mathbf{M} \in \mathfrak{C}$, then $\mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq)$ is contained in the modal logic of assertions valid in $\mathscr{F}$ at w_0.

Suppose now that we have managed to show that $\mathsf{L} \subseteq \mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ using the strategy for lower bounds, and that $\mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{L}$ using the method for upper bounds. Then we obtain:

$$\mathsf{L} \subseteq \mathsf{ML}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{L}$$

and so $\mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq) = \mathsf{L}$.

This method provides a general strategy that can be applied to characterize $\mathsf{ML}_{\mathscr{L}}(\mathfrak{C}, \sqsubseteq)$ in terms of other well-known modal logics and it can be applied in principle to whatever class of structures we want. It could therefore be interesting to consider certain classes of algebras (for example some varities or quasivarieties) and interesting binary relations on them in order to find the modal logic they give rise to: this is exactly the framework in which the work presented in [2] lives.

The authors consider the class, which is actually a variety, of abelian groups AG together with the relation given by the operator $\mathbf{IS}$: for $\mathbf{A}, \mathbf{B} \in \mathsf{AG}$, the notation $\mathbf{A} \leq \mathbf{B}$ will stand for $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, i.e. $\mathbf{A}$ is isomorphic to a subgroup of $\mathbf{B}$. They manage to show that the modal logic $\mathsf{ML}(\mathsf{AG}, \leq)$ is exactly $\mathsf{S4.2}$

The fact that $\mathsf{S4.2}$ is a lower bound for the modal logic of abelian groups is clear, since it is straighforward to prove that the relation $\leq$ on AG is reflexive, transitive and directed, since given $\mathbf{A}, \mathbf{B} \in \mathsf{AG}$ there exists a common upper bound for them in terms of $\leq$, the cartesian product $\mathbf{A} \times \mathbf{B}$ in which both $\mathbf{A}$ and $\mathbf{B}$ can be embedded.

In order to prove that $\mathsf{S4.2}$ is also an upper bound for the modal logic of abelian groups, the strategy of finding a labeling of certain frames which are complete with respect to $\mathsf{S4.2}$ is used. Without entering into detail, it turns out that often the existence of a labeling can be broken down into simpler statements; the control statements can be seen as building blocks through which we can construct more complex statements and therefore labelings (see [3] for more details). The authors prove that given any abelian group $\mathbf{A}$, there is always the possibility of building another group $\mathbf{B}$ which satisfies exactly some specific control statements and in which $\mathbf{A}$ can be embedded, i.e. $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$. In this way they show that $\mathsf{S4.2}$ is an upper bound for $\mathsf{ML}(\mathsf{AG}, \leq)$.

The construction that appears in [2], which is based on the tools of localisations and controlled group amplifications, uses at great extent the fact that the groups are abelian: for example with non-abelian groups, there wouldn't be the possibility of dealing with controlled group amplifications, since they would not be well defined. In fact the authors leave the readers with an open question, which is related to what happens in the case of non-abelian groups.

It turns out that a different construction can be used in the general case of groups (even non-abelian ones) and, since this construction does not use the operation of inverse, it works for monoids as well. Moreover, this construction uses the same control statements as the ones that were introduced for abelian groups: this is not obvious, since control statements are defined as first order sentences in the language of the theory we are considering, which satisfy some specific axiom schemata (see [3] for more details). This means that in principle any set of formulas of the language can be chosen, provided that all the formulas in the collection satisfy some specific properties. Therefore, even if a construction does not work for a certain class and for a specific set of control statement, there may be another set of control statements and/or another construction that work for that class of structures.

In this case there is no need to look for other control statements, since the ones provided in [2] work equally well in the case of monoids, as long as another construction, which does not require commutativity and existence of inverse, is used. Therefore, using the same control statements as the ones introduced in [2] but a different construction, it can be shown that $\mathsf{ML}(\mathsf{M}, \leq) = \mathsf{S4.2}$, where we denoted by M the variety of monoids.

We highlight that this result is original and it constitutes an extension of the one presented in [2], in fact any monoid homomorphism between two groups is also a group homomorphism; suppose that $\mathbf{A}$ and $\mathbf{B}$ are groups and that $\mathbf{A}' \subseteq \mathbf{A}$ and $\mathbf{B}' \subseteq \mathbf{B}$ are monoids, then if $f : \mathbf{A}' \to \mathbf{B}'$ is a monomorphism, the natural extension of f to $\mathbf{A}$ is a group monomorphism from $\mathbf{A}$ to $\mathbf{B}$. Using our notation, if $\mathbf{A}' \leq \mathbf{B}'$ in the sense of monoids, then $\mathbf{A} \leq \mathbf{B}$ in the sense of groups. Because our purpose was to characterize the modal logic induced by the relation $\leq$ (ore equivalently by the operator $\mathbf{IS}$), this observation witnesses that the result above really extends the one about groups.

We observe that this problem could have many future developments: it could be natural to analyze what happens for other classes of structures with the relation given by $\mathbf{IS}$, or we could even switch to other significant operators, like $\mathbf{SP}$, $\mathbf{HSP}$, $\mathbf{ISP}_u$ and so on. Let's consider for example the case in which the class is a given variety V and the relation $\leq$ is induced by the operator $\mathbf{IS}$ as before: if V does not have the joint embedding property the relation is not directed and therefore it is not true that $\mathsf{S4.2}$ is a lower bound for $\mathsf{ML}(\mathsf{V}, \leq)$. However, $\leq$ is clearly reflexive and transitive whatever the variety V is, which yields $\mathsf{S4} \subseteq \mathsf{ML}(\mathsf{V}, \leq)$: this is what happens for example for lattices. Hence, one of the many possible further directions of this work could be the one of finding conditions on V which allow us to characterize the modal logic of the embedding on V in terms of modal logics which are based on $\mathsf{S4}$.

Finally we remark that the problem we deal with is different from the one that is presented in [4]: there the authors work on the collection of all the models of a fixed language and investigate the modal logic of this class with respect to the relation of being a submodel. In other words, from an algebraic point of view, they consider the class $\mathfrak{C}$ consisting of all the algebras of the same fixed type together with the relation $\geq$ such that given $\mathbf{A}, \mathbf{B} \in \mathfrak{C}$,

$\mathbf{A} \geq \mathbf{B}$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$; then they try to characterize $\mathsf{ML}(\mathfrak{C}, \geq)$. This approach is different from ours mainly for two reasons: firstly, we don't work with the class of all algebras of the same type (which is very wide and contains structures that may be very different from each other), but we restrict to classes of algebras of the same type satisfying some specific axioms. Moreover the relation considered in [4] is exactly the opposite with respect to the one we deal with: for us $\mathbf{A}$ in in relation with $\mathbf{B}$ if $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, while for the authors of [4] $\mathbf{A}$ is in relation with $\mathbf{B}$ if $\mathbf{B} \in \mathbf{IS}(\mathbf{A})$.

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Medvedev Logic and Combinatorial Geometry

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Medvedev logic, or the *logic of finite problems*, is a well-known intermediate logic first introduced by the Russian mathematician Medvedev in his 1963 article [10]. It may be semantically defined as the logic of the Kripke frames $\{(\mathrm{Sub}n \smallsetminus \{\emptyset\}, \supseteq)\}_{n\in\mathbb{N}}$, i.e. the powersets Sub$n$ of finite non-empty sets ordered by reverse inclusion, with the empty subset removed. For background and references on Medvedev logic we refer to [5]. By the *Medvedev variety* we mean the subvariety of Heyting algebras corresponding to Medvedev logic, i.e. the closure under homomorphic images, subalgebras, and products of the Heyting algebras of upper-closed subsets of the posets $\{(\mathrm{Sub}n \smallsetminus \{\emptyset\}, \supseteq)\}_{n\in\mathbb{N}}$.

Connections between Medvedev logic and cellular structures—notably, simplicial complexes—have long been known among specialists.[1] Indeed, the Medvedev frame $(\mathrm{Sub}n \smallsetminus \{\emptyset\}, \supseteq)$ is the poset of faces of an n-dimensional simplex ordered by reverse inclusion. The aim of this contribution is to initiate a systematic investigation of such connections. We discuss here two categories of central importance in combinatorial geometry, finite simplicial complexes and simplicial sets; in the manuscript [3], currently in preparation, we offer a more extensive treatment including, among others, ordered and infinite complexes, Δ-sets, and symmetric simplicial sets. It is to be hoped that these semantics based on combinatorial geometry may eventually become a further tool to tackle questions about Medvedev logic, which is notoriously difficult to analyse.

Simplicial Complexes

A classical treatment of simplicial complexes is [2]; for a contemporary account, see e.g. [6].

A (*finite*) *simplicial complex* Σ on the *set of vertices* V is a set of subsets of the finite set V such that the following conditions are satisfied.

1. Each member of Σ is non-empty.
2. For each $v \in V$, $\{v\} \in \Sigma$.
3. For each $\sigma \in \Sigma$ and for each $\emptyset \neq \tau \subseteq \sigma$, $\tau \in \Sigma$.

[1]The third-named author would like to record his gratitude to Valentin Shehtman for having shared with him his extensive knowledge of the literature on Medvedev logic on the occasion of a visit to the University of Milan.

Let us write vrtΣ for the set of vertices of Σ. For simplicial complexes Σ and Δ, a *simplicial map* $\Sigma \to \Delta$ is a function $f\colon \mathrm{vrt}\Sigma \longrightarrow \mathrm{vrt}\Delta$ such that $f[\sigma] \in \Delta$ holds for each $\sigma \in \Sigma$, where $f[-]$ denotes the direct image along f. Simplicial complexes and simplicial maps form a category S.

Subobjects of an object Σ of S are known as *subcomplexes* of Σ; they may be identified with the simplicial complexes Δ on some set of vertices $W \subseteq \mathrm{vrt}\Sigma$ such that $\delta \in \Sigma$ for each $\delta \in \Delta$. We write SubΣ for the set of subcomplexes of the complex Σ. It is elementary that SubΣ under inclusion order is a finite distributive lattice with top Σ and bottom the empty subcomplex. It follows SubΣ has a unique structure of Heyting algebra. Write M for the full subcategory of Heyting algebras on those algebras isomorphic to SubΣ for some simplicial complex Σ. The subobject functor

$$\mathrm{Sub}\colon \mathsf{S} \longrightarrow \mathsf{M}^{\mathrm{op}} \tag{0.1}$$

acts contravariantly on simplicial maps $f\colon \Sigma \to \Delta$ by inverse images (pullback of subobjects along f) in the standard manner. It is not difficult to prove Sub is part of a dual equivalence of categories. The explicit description of the adjoint $\mathsf{M}^{\mathrm{op}} \to \mathsf{S}$, also not difficult, is conceptually interesting in that it features the representation of simplicial complexes as a category of finite spaces and open maps; we omit details for reasons of space, and state our first result as:

Theorem 1. *The functor (0.1) is part of a dual equivalence of categories between S and M. Moreover, the variety generated by the class of objects of M is the Medvedev variety, and so the logic of the category S of finite simplicial complexes is Medvedev logic.*

Presheaf Toposes

For background on topos theory, and on presheaf toposes in particular, please see e.g. [8]. A presheaf category is a category whose objects are presheaves on a small category C, i.e. contravariant functors from C to Set, and whose maps are natural transformations between them. We denote by $\widehat{\mathsf{C}}$ the category of presheaves on C. A subpresheaf G of F in $\widehat{\mathsf{C}}$ is a subobject in the presheaf category, namely a class, up to isomorphism, of a monomorphism from G to F. Since a presheaf category turns out to be an elementary topos, it is also referred to as a presheaf topos.

A general fact concerning toposes is that, given an arbitrary topos $\mathcal{E}$ and an object X in it, the set of subobjects Sub$X$ of X can be equipped with a natural structure of Heyting algebra. In case the topos is a presheaf category, for every presheaf F the Heyting algebra Sub$F$ is complete, being the identity on F the top and the natural transformation from the terminal functor to F the bottom.

We will provide a criterion that is helpful in identifying the intermediate logic determined by certain presheaf toposes. By definition, the intermediate logic determined by an arbitrary topos $\mathcal{E}$ is the logic uniquely associated with the variety generated by the class of Heyting algebras Sub$X$, as X ranges over all objects of $\mathcal{E}$. This coincides with the intermediate logic of all fomulæ valid in the internal Heyting algebra structure of the subobject classifier of $\mathcal{E}$, though we will not detail this fact here.

For presheaves, matters simplify. By standard general theory, the intermediate logic of a presheaf topos $\widehat{\mathsf{C}}$ is determined by subobjects of the representable functors only, i.e.

by the Heyting algebras $\mathrm{Sub}(\hom(-,X))$; the elements of such algebras may in turn be identified with sieves on the object X. Building on this, Ghilardi in [7] showed how to construct out of the slice categories C/X a Kripke frame whose logic coincides with that of $\widehat{\mathsf{C}}$.

Moreover, we observe that if C admits a specific factorisation system then the Heyting algebras of subobjects of representable functors are determined by the posets of subobjects in the site C. Recall a *split epimorphism* (also called a *retraction*) in a category is an arrow $r\colon a \to b$ such that there exists a *section* $s\colon b \to a$ with $r \circ s$ the identity on b. The category C has (*split epi/mono) factorisations* if each arrow in C factors as a split epimorphism followed by a monomorphism. We prove:

Theorem 2. *Let* C *be a small category with (split epi/mono) factorisations. Then for every object* X *there is an isomorphism of Heyting algebras*

$$\mathrm{Sub}(\hom(-,X)) \cong \mathrm{Down}(\mathrm{Sub}X). \qquad (*)$$

The right-hand side of (*) denotes the Heyting algebra of all downward-closed subsets of the poset $\mathrm{Sub}X$.

Hence, under the hypotheses of Theorem 2, the intermediate logic of a presheaf topos $\widehat{\mathsf{C}}$ is the logic of the opposites of the posets of subobjects $(\mathrm{Sub}X)^{\mathrm{op}}$, as X ranges over objects of C.

Simplicial Sets

We finally turn to the presheaf topos of simplicial sets. Any simplicial complex equipped with a partial order of its vertices that is linear on each simplex determines a simplicial set in a natural manner. In this sense simplicial sets provide a generalisation of (ordered) simplicial complexes. In fact, simplicial sets are considerably more general than simplicial complexes under several respects. Nonetheless, the intermediate logic of the presheaf topos of simplicial sets is once again Medvedev logic. For background on simplicial sets we refer e.g. to [9, 6, 8].

The category SSet of simplicial sets is the presheaf topos on the simplex category $\triangle$, namely the category of finite non-empty ordinals with morphisms the monotone functions. We denote by $[n]$ the object of $\triangle$ given by the totally ordered set with $n+1$ elements.

It is well known, and not hard to prove, that the simplex category $\triangle$ admits (split epi/mono) factorizations (for details see e.g. [9]), so Theorem 2 applies. As a consequence, we infer the intermediate logic of simplicial sets is the one determined by the posets $(\mathrm{Sub}[n])^{\mathrm{op}}$, as n ranges over non-negative integers. But subobjects of $[n]$ (equivalence classes of monomorphisms in $\triangle$ with codomain $[n]$) are uniquely determined by their set-theoretic images, which are the non-empty subsets of a set with $n+1$ elements.

These considerations lead to the following theorem:

Theorem 3. *The intermediate logic determined by the presheaf topos* SSet *is Medvedev logic.*

Remark 1. In [1] and [4], as well as in a further forthcoming paper, the third-named author introduced and studied in collaboration with several co-authors the intermediate logic associated to classes of compact polyhedra. Without entering details, it is important to emphasise that in the framework of that research the logic determined by a simplicial complex is defined as the logic of the poset of simplices ordered by *inclusion*, as opposed to the *reverse* inclusion adopted in the present abstract. The overall picture then changes altogether. For example, it is proved in [4] that the logic of all simplicial complexes (under *inclusion* of simplices) is full intuitionistic logic, in stark contrast to Theorem 1.

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A Dual Proof of Blok's Lemma

Rodrigo Nicolau Almeida, Nick Bezhanishvili, and Antonio M. Cleani

Blok-Esakia Theorem

A *modal companion* of a superintuitionistic (si) logic **L** is any normal modal logic above **S4** in which **L** fully and faithfully embeds via the Gödel translation. The notion of a modal companion has a rich theory [6, 8, 10, 7, 4, 12, 3], culminating in a result known as the *Blok-Esakia theorem* [7, 2]. The latter states that the lattice si logics is completely isomorphic to the lattice of normal extensions of **Grz**, via the mapping that sends a si logic **L** to the least normal extension of **Grz** containing the Gödel translations of all theorems of **L**.

The literature contains several proofs of the Blok-Esakia theorem. Blok's original proof [2] is algebraic and notoriously involved (see also [11]). Esakia appears to have given a dual proof, which remains unpublished. Zacharyaschev later gave a proof using the machinery of canonical formulas [13], which Jeřábek [9] later extended to rule systems using canonical rules. More recently, Bezhanishvili [1] (see also [5]) offered an alternative proof based on *stable* canonical rules, which are algebra-based rules that are to filtration what Jerabek's canonical rules are to selective filtration.

Our contribution is to show that the proof by Bezhanishvili and Cleani can be carried out without the machinery of stable canonical rules. Moreover, the key idea of that proof can be adapted to obtain a dual, order-topological proof that resembles Blok's original algebraic one, and gives some intuitions on it.

Blok's Lemma

We begin with some preliminary definitions. Let $\mathcal{H}$ be a Heyting algebra. The modal algebra $\sigma\mathcal{H}$ is constructed by expanding the free Boolean extension $B(\mathcal{H})$ of $\mathcal{H}$ with the operator

$$\Box a := \bigvee\{b \in H : b \leqslant a\}.$$

It is well known that $\sigma\mathcal{H}$ is always a **Grz**-algebra.

Conversely, given an **S4**-algebra $\mathcal{M}$, the *skeleton* $\rho\mathcal{M}$ of $\mathcal{M}$ is simply the Heyting algebra of open elements of $\mathcal{M}$. We recall that an element a of $\mathcal{M}$ is open when $\Box a = a$, and that the Heyting implication of $\rho\mathcal{M}$ is given by $a \to b := \Box(a \to b)$.

Dually, when $\mathcal{X}$ is an Esakia space we let $\sigma\mathcal{X} := \mathcal{X}$. Conversely, when $\mathcal{Y}$ is an **S4**-space, we let $\rho\mathcal{Y}$ be the Esakia space that results from $\mathcal{Y}$ by collapsing all clusters, and endowing the result with the quotient topology under the cluster collapse mapping. The algebraic and topological versions of the mappings σ, ρ are dual to one another.

A **Grz**-algebra $\mathcal{M}$ is called *skeletal* when it is isomorphic to $\sigma\rho\mathcal{M}$. Blok derives the Blok-Esakia theorem as a consequence of the following Lemma, now widely known as *Blok's Lemma*.

Lemma 1 (Blok's Lemma)**.** *Let $\mathscr{M}$ be a* **Grz**-*algebra. Then $\mathscr{M} \in \mathsf{ISP_U}(\sigma\rho\mathscr{M})$.*

Let $\mathscr{M}, \mathscr{N}$ be modal algebras and let $\mathscr{A}, \mathscr{B}$ be Boolean subalgebras of $\mathscr{M}, \mathscr{N}$ respectively. A mapping $h : \mathscr{A} \to \mathscr{B}$ is called a $\Box$-*homomorphism* when it is a Boolean homomorphism and $h(\Box a) = \Box h(a)$ whenever $\Box a \in A$. The key step in Blok's proof of Lemma 1 is the following result.

Lemma 2 (Algebraic embedding lemma)**.** *Let $\mathscr{M}$ be a* **Grz**-*algebra and let $\mathscr{N}$ be a finite Boolean subalgebra of $\mathscr{M}$. Then there is a $\Box$-embedding $h : \mathscr{N} \to \sigma\rho\mathscr{M}$.*

The proof given in [1], in turn, makes use of stable maps between modal spaces:

Definition 1. Given modal spaces $\mathscr{X} = (X, R)$ and $\mathscr{Y} = (Y, R)$, a continuous function $f : \mathscr{X} \to \mathscr{Y}$ is said to be *stable* if $f(x)Rf(y)$ holds whenever xRy. Given $D \subseteq \mathsf{Clop}(\mathscr{X})$ we say that f satisfies the *bounded domain condition* (BDC) with respect to D if for all $U \in D$, if $f(x)Ry$ and $y \in U$, then there is some x' such that xRx' and $f(x') \in U$.

Given Y a finite **S4**-space, and D a domain, $\mathscr{J}(Y, D)$ denotes the *stable canonical rule.* Bezhanishvili and Cleani show that for a **Grz**-space X, $X \not\Vdash \mathscr{J}(\mathscr{Y}, D)$ implies $\rho X \not\Vdash \mathscr{J}(\mathscr{Y}, D)$, which amounts to showing the following lemma:

Lemma 3. *Given a* **Grz**-*space $\mathscr{X} = (X, R)$, a finite* **S4**-*space $\mathscr{Y} = (Y, R)$, and a surjective stable map $p : \mathscr{X} \to \mathscr{Y}$ satisfying the BDC for a domain D, there is a stable surjection $p' : \rho\mathscr{X} \to \mathscr{Y}$ satisfying the BDC for the domain D.*

One can then show the following:

Proposition 1. *The following statements are equivalent:*

1. *Blok's Lemma;*
2. *The algebraic stable embedding lemma;*
3. *The stable surjection lemma.*

Step-by-Step Proofs of Blok's Lemma

By considering the specifics of the algebraic proof of Blok's lemma and the order-topological one of [1], we will obtain a new dual proof which closely mirrors the original one of Blok. This starts with the following:

Lemma 4 (Algebraic one-step embedding)**.** *Let $\mathscr{M}$ be a* **Grz**-*algebra and let $\mathscr{N}$ be a finite Boolean subalgebra, such that $h : \mathscr{N} \to \sigma\rho\mathscr{M}$ is a $\Box$-embedding fixing all open elements. If $x \in \mathscr{M}$ is arbitrary, then there are finitely many open elements C, and a $\Box$-embedding $h' : \langle \mathscr{N} \cup \{x\} \cup C\rangle \to \sigma\rho\mathscr{M}$ such that $h' \restriction_{\mathscr{N}} = h$.*

Indeed, having this lemma, and an arbitrary finite Boolean subalgebra $\mathscr{N}$, one enumerates $\mathscr{N} = \mathscr{N}_{op} \cup \{x_1, ..., x_n\}$ where $\mathscr{N}_{op} = \langle\{\Box a : \Box a \in \mathscr{N}\}\rangle$. Then one sets $\mathscr{N}_0 = \mathscr{N}_{op}$, and successively adds one element, creating a sequence of algebras:

$$\mathscr{N}_0 \to \mathscr{N}_1 \to ... \to \mathscr{N}_n$$

where $\mathscr{N} \subseteq \mathscr{N}_n$. Having a $\Box$-embedding of the latter algebra gives us a $\Box$-embedding of $\mathscr{N}$. It is the way that such elements are picked which demands the **Grz**-axiom: given $\mathscr{N}_k$, to extend the embedding to $\langle \mathscr{N}_k \cup \{x_{k+1}\}\rangle$ one needs only define the image of x_{k+1}. This is done by picking, for each $c \in \mathscr{N}_k$, an element element w_c which is defined, by letting $u_c = \neg x_{k+1} \vee c$ as:

$$w_c := \neg(\Box(u \to \Box u) \to \Box u).$$

The use of the **Grz**-axiom lies in ensuring that $\Box(\Box(u \to \Box u) \to \Box u) \leqslant u$; dually, this follows if:

$$u \leqslant \Diamond \pi u$$

where $\pi u = u \wedge \neg\Diamond(\Diamond u \wedge \neg u)$. This is called by Esakia the *rest* of u; dually, given a clopen U, πU is called the set of *passive points of* U, and it is known that the **Grz**-axiom corresponds to every point in U being below a passive point.

Applying the key idea of the proof of [1, Main Lemma], one can obtain a dual step-by-step proof of Blok's lemma. Given a space $\mathscr{X}$, let $\rho : \mathscr{X} \to \rho\mathscr{X}$ be the cluster collapse continuous map. Moreover, a surjection $\varrho : \mathscr{X} \to \mathscr{Y}$ is called a *cluster-reducing map* when it is the quotient map induced by some equivalence relation on $\mathscr{Y}$ that never relates points belonging to different clusters in $\mathscr{X}$.

Lemma 5 (Dual one-step surjection)**.** *Let* $\mathscr{X} = (X, R), \mathscr{Y} = (Y, R)$ *and* $\mathscr{Y}' = (Y', R)$ *be* **S4** *spaces with the following properties.*

- $\mathscr{X} = (X, R)$ *is a* **Grz** *space;*
- $Y' = Y \sqcup \{\bullet\}$ *and there is a cluster-reducing map* $\varrho : \mathscr{Y}' \to \mathscr{Y}$ *which identifies* $\bullet$ *with some point in its cluster, but does not identify any other points;*
- *There is a stable surjection* $f : \mathscr{X} \to \mathscr{Y}'$ *satisfying the BDC for some* $D \subseteq Y'$*;*
- *There is a stable surjection* $g : \sigma\rho\mathscr{X} \to \mathscr{Y}$ *satisfying the BDC for* $\varrho[D]$.

Then there is a stable map $h : \sigma\rho\mathscr{X} \to \mathscr{Y}'$ *satisfying the BDC for* D.

We now sketch the main idea of the proof. Let $x \in Y'$ be unique with $\varrho(x) = \varrho(\bullet)$. If $y \notin \{x, \bullet\}$, we can set $h^{-1}[y] = g^{-1}[\varrho(y)]$. Note $\max(f^{-1}[\bullet])$ and $\max(f^{-1}[x])$ are disjoint and closed. Moreover, because $\mathscr{X}$ is a **Grz**-space, both these sets consist entirely of passive elements. Consequently, their images $U_\bullet := \rho[\max(f^{-1}[\bullet])]$ and $U_x := \rho[\max(f^{-1}[x])]$ are closed in $\sigma\rho\mathscr{X}$. Now, $U_\bullet$ and U_x are contained in the clopen $U := \rho(f^{-1}\{x, \bullet\})$, thus we can partition U in two clopens $V_x \supseteq U_x$ and $V_\bullet \supseteq U_\bullet$. We then put $h^{-1}(x) = V_x$ and $h^{-1}(\bullet) = V_\bullet$. So h is a stable surjection that satisfies the BDC for D, as desired.

Having obtained Lemma 5, we prove Blok's lemma following the algebraic proof strategy. Given a finite **S4**-space $\mathscr{Y} = (Y, R)$, we set $\mathscr{Y}_0 = \sigma\rho\mathscr{Y}$ and form the inverse chain

$$\mathscr{Y}_0 \leftarrow \mathscr{Y}_1 \leftarrow \ldots \leftarrow \mathscr{Y}_n$$

where $\mathscr{Y}_n = \mathscr{Y}$, and $\mathscr{Y}_{i+1}$ is obtained as some cluster-expansion of $\mathscr{Y}_i$ by one additional element; the existence of the surjections in the chain being guaranteed by Lemma 5. The algebra $\mathscr{N}_0$ is the dual of $\mathscr{Y}_0$, and the posets $\mathscr{Y}_k$ – whilst not being isomorphic to the dual of $\mathscr{N}_k$ – can be seen as refinements of the decomposition given by the algebraic construction.

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Language for Crash Failures in Impure Simplicial Complexes

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Simplicial complexes are a well-known semantic framework in combinatorial topology to model synchronous and asynchronous distributed systems. A common type of faults considered in synchronous computation is *crash failures*. In a system with crash failures, each live process may be uncertain regarding which of the other processes have already crashed. In simplicial complexes, this is modeled semantically by considering so-called *impure* simplicial complexes. In this extended abstract, we discuss which object language is appropriate and expressive enough to reason about synchronous distributed systems with crash failures using the impure simplicial semantics.

Epistemic logic investigates knowledge and belief, and change of knowledge and belief, in multi-agent systems [16, 9, 5]. Knowledge change was extensively modeled in temporal epistemic logics [20, 13, 8] and in dynamic epistemic logics [1, 6, 19]. Epistemic logical semantics is often based on *Kripke models*, that consist of an abstract domain of global states, or worlds, between which binary relations of accessibility (or indistinguishability, depending on the agents' epistemic strength) are defined, one for each agent [17].

Combinatorial topology [14] has been used in distributed computing to model concurrency and asynchrony since [10, 18, 3], including higher-dimensional topological properties [15, 22]. Geometric manipulations such as subdivision have natural combinatorial counterparts. *Simplicial models* consist of an abstract set of *vertices* representing agents' local states. These agent-colored vertices are combined into sets called *simplices*, with a standard chromatic restriction that each simplex contain no more than one vertex per agent. Global states of the system correspond to those simplices that are maximal with respect to set inclusion and are called *facets*. *Pure* simplicial complexes correspond to distributed systems without crashes, hence, require that each facet contain exactly one vertex for each of the agents. Crashed agents are modeled by allowing facets to have fewer vertices than the total number of agents, with the understanding that all agents missing from a facet are *dead*, i.e., have crashed, whereas all agents present in the facet (as a single vertex) are *alive*. The collection of sets of vertices (simplices) in a given simplicial model is assumed to be downward closed with respect to set inclusion, with the exception of the empty set. Proper subsets of any facet are called *faces* and can be viewed as partial global states of the system.

In lieu of giving a lengthy formal definition [7], in Fig. 1 we provide examples of one pure ($\mathscr{C}_1$) and two impure ($\mathscr{C}_2$ and $\mathscr{C}_3$) simplicial models for a distributed system with three agents a, b, and c:

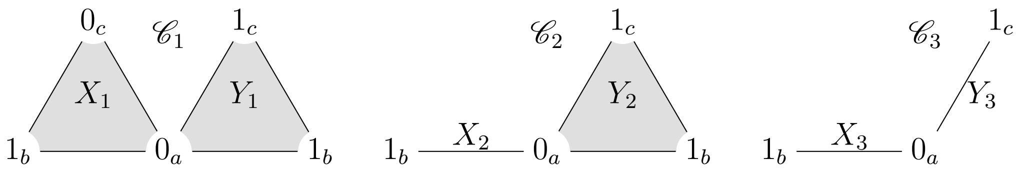

Figure 1: Impure and pure simplicial models

Each model $\mathscr{C}_i$ consists of two facets X_i and Y_i (global states) that agent a cannot distinguish, as evidenced by its vertex (local state) 0_a belonging to both. Model $\mathscr{C}_1$ is pure because its two facets (two gray triangles) X_1 and Y_1 consist of three vertices (one per agent) each. Thus, a is sure that all agents are alive and knows the value of b's variable as it is true (depicted as 1_b) in both X_1 and Y_1. On the other hand, a does not know the truth value of c's variable as it is false (0_c) in X_1 and true (1_c) in Y_1. Models $\mathscr{C}_2$ and $\mathscr{C}_3$ are impure because each contains at least one facet with strictly less than three agents: agent c is dead in X_2 of $\mathscr{C}_2$ and in X_3 of $\mathscr{C}_3$, and, additionally, agent b is dead in Y_3 of $\mathscr{C}_3$. Note that facets X_2, X_3, and Y_3 in the impure models $\mathscr{C}_2$ and $\mathscr{C}_3$ are edges that can also be found in the pure model $\mathscr{C}_1$. However, there the corresponding edges are sides of triangles, or in simplicial complex terms, are faces of larger facets X_1 and Y_1, without being facets themselves. In both $\mathscr{C}_2$ and $\mathscr{C}_3$, agent a is unsure whether c is alive (and, additionally, whether b is alive in $\mathscr{C}_3$).

Note that we have already smuggled a small change from the standard logical language in the form of local propositional variables p_a, p_b, q_b, etc. They originate from the notion of an agent's local state in distributed systems, which is always known by the agent. Thus, a propositional variable p_a pertaining to the local state of a should be known by a, as formalized by the *locality axiom* $K_a p_a \vee K_a \neg p_a$ where modality K_a represents agent a's knowledge [11, 7]. Local variables represent a natural but not the only choice. A logic of impure simplicial complexes with standard propositional variables that are unattached to any agent (global) can be found, e.g., in [12].

We believe that a proper logic for distributed systems should include both types of variables: local variables for describing agents' local states and global variables describing global properties of the system that need not be known to any agent. For instance, asynchronous systems are typically modeled to have global time that no agent has access to, making this global time a good example of a global variable that does not belong to any agent and is, generally, not known by any agent. Logically, this would be realized by applying the locality axiom to local variables only.

The dichotomy of local and global variables is not the only choice that has been considered. Another non-trivial question regards the effect agents' crashes have on the knowledge of live agents, in particular, on their knowledge of the local variables of crashed agents. Consider again impure models $\mathscr{C}_2$ and $\mathscr{C}_3$ in Fig. 1. Does agent a know the value of, say, b's variable p_b there? The only obvious answer is that the value of p_b is known in $\mathscr{C}_2$ as it is true in both X_2 and Y_2.

But what happens with p_b in facet Y_3 of model $\mathscr{C}_3$? And what does a know about it in facet X_3? Were p_b a global variable, as in [12], its truth value would have been determined by the whole facet Y_3, and the crash of agent b would not affect it. On the other hand, there is no universally acceptable way of assigning a truth value to a local variable p_b in facet Y_3. This prompted the introduction of the third truth value 'undefined' in [7]. Propositionally, this value is treated according to the 3-valued Weak Kleene Logic, with the undefined value "infecting" any propositional formula it participates in. The question about knowledge in presence of undefined values is more subtle. In global state X_3 of model $\mathscr{C}_3$, given that p_b is undefined in Y_3, (i) should a know p_b to be true based on X_3 alone, the sole facet where p_b is defined or (ii) should a not know p_b to be true because it is not true in Y_3, which a considers possible? Both options may seem reasonable at first but option (ii) has an undesirable consequence for the dual modality $\widehat{K}_a := \neg K_a \neg$, which stands for a considers it possible. Indeed, if $\mathscr{C}_3, X_3 \not\models K_a p_b$ according to (ii), then $\mathscr{C}_3, X_3 \models \widehat{K}_a \neg p_b$, i.e., agent a would have consider it possible that p_b is false despite it not being false in any facet of $\mathscr{C}_3$. This consideration explains why option (i) was chosen in [7]. It should be noted that the resulting logic is different from the way modalities work in [4].

The resulting epistemic logic of impure simplicial complexes, based on 3-valued Weak Kleene Logic on the propositional level and with local variables only, was axiomatized in [21]. The difficulty was that, as we soon discovered [2], it did not satisfy the Hennessy–Milner property for the natural notion of bisimulation. Worse than that, we have shown that no reasonable local definition of bisimulation relying on the standard back-and-forth relations would have Hennessy–Milner [21].

A failure of Hennessy–Milner often means that the language is not expressive enough. And the property lacking expressivity in terms of local variables only was quite obvious. Above, while we used the term "know", corresponding to the K_a modality for local variables, we resorted to "is sure that" regarding agents being alive or dead. The reason for this was that the latter was not expressible in the language with local variables only [2]. Hence, using "know" would have been misleading. Since one of the objectives in a distributed systems with crash failures is to reason in presence of crash failures, a language not expressive enough to talk about these crash failures in the object language is suboptimal.

Thus, based both on the desired applications and on the logical evidence of insufficient expressivity, we believe that the object language for the logic of impure simplicial complexes should include both local and global variables and that these global variables should, at the minimum, include atoms expressing that a particular agent is alive. In [2], we have shown that the logic with such atoms a for each agent a does indeed possess the Hennessy–Milner property. We are currently preparing for submission a manuscript with a complete axiom system for this logic, which extends that from [21] for local variables only.

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Free algebras and coproducts in varieties of Gödel algebras

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Free Heyting algebras play a fundamental role in the study of the intuitionistic propositional calculus IPC because they arise as Lindenbaum-Tarski algebras, whose elements are equivalence classes of propositional formulas over a fixed set of variables modulo logical equivalence in IPC. Esakia duality (see [10]) proved to be a powerful tool for understanding the structure of free Heyting algebras, which are notoriously difficult to describe. Recall that a Stone space is a topological space that is compact, Hausdorff, and has a basis consisting of clopen (i.e., closed and open) subsets.

Definition 1. An *Esakia space* is a Stone space X equipped with a partial order $\leq$ such that

(1) ${\uparrow}x := \{y \in X : x \leq y\}$ is closed for every $x \in X$,

(2) ${\downarrow}V := \{x \in X : x \leq y \text{ for some } y \in V\}$ is clopen for every $V \subseteq X$ clopen.

Every Esakia space X gives rise to the Heyting algebra $\mathsf{ClopUp}(X)$ of the clopen upsets of X ordered by inclusion, where $U \subseteq X$ is an *upset* if ${\uparrow}x \subseteq U$ for every $x \in U$. Vice versa, the prime spectrum $\mathsf{Spec}(H)$ of a Heyting algebra H, which is the set of the prime filters of H, becomes an Esakia space once ordered by inclusion and suitably topologized. This correspondence extends to Heyting homomorphisms between Heyting algebras and continuous p-morphisms between Esakia spaces, where $f\colon X \to Y$ is a p-morphism if $f[{\uparrow}x] = {\uparrow}f(x)$ for every $x \in X$.

Theorem 1 (Esakia duality)**.** *The category of Heyting algebras and Heyting homomorphisms is dually equivalent to the category of Esakia spaces and continuous p-morphisms.*

Different methods to study the Esakia duals of free Heyting algebras have been developed. Universal models, first investigated in [19, 4], describe the points of finite depth of the Esakia duals of finitely generated free Heyting algebras (see, e.g., [5, Sec. 3]). A different approach, known as the step-by-step method and developed in [20, 12], builds the Esakia duals of finitely generated free Heyting algebras as the inverse limits of systems of finite posets. This approach has been recently generalized beyond the finitely generated setting [2]. However, due to the complexity of free Heyting algebras, obtaining a tangible and complete description of their Esakia duals seems difficult—if not impossible—particularly for those that are free over infinitely many generators. This naturally leads us to consider free algebras in subvarieties of the variety of Heyting algebras. We will turn our attention to free Gödel algebras.

Definition 2. A *Gödel algebra* is a Heyting algebra satisfying the prelinearity axiom $(x \to y) \vee (y \to x) = 1$.

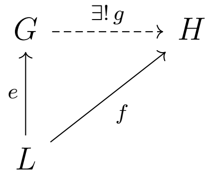
The variety GA of Gödel algebras is generated by the totally ordered Heyting algebras and provides the algebraic semantics for the superintuitionistic propositional logic known as the Gödel-Dummett logic [8]. This logic has attracted much attention, partly because it can also be regarded as a fuzzy logic (see, e.g., [3] and [18, Sec. 4.2]).

It is well known that Esakia duality restricts to a duality for Gödel algebras. An Esakia space X is called an *Esakia root system* if the order $\leq$ on $\uparrow x$ is total for every $x \in X$.

Proposition 1. *The category of Gödel algebras and Heyting homomorphisms is dually equivalent to the category of Esakia root systems and continuous p-morphisms.*

As all finitely generated Gödel algebras are finite [16], GA is a locally finite variety. The Esakia duals of finitely generated free Gödel algebras were described in [13], while the Esakia duals of Gödel algebras free over finite distributive lattices[1] were described in [1].

Definition 3. A Gödel algebra G is said to be free over a distributive lattice L via a lattice homomorphism $e \colon L \to G$ when the following holds: for every Gödel algebra H and lattice homomorphism $f \colon L \to H$, there is a unique Heyting homomorphism $g \colon G \to H$ such that $g \circ e = f$.



Our main result generalizes the descriptions of [1] beyond the finitely generated setting by providing a dual description of Gödel algebras free over distributive lattices, without any restriction on the cardinality of the lattice. As a consequence, we obtain a dual description of free Gödel algebras over any set of generators that generalizes the one of [13]. To provide such a description, we first recall Priestley duality for distributive lattices (see, e.g., [11]).

Definition 4. A *Priestley space* is a Stone space X equipped with a partial order $\leq$ satisfying the *Priestley separation axiom*: if $x, y \in X$ with $x \not\leq y$, then there is a clopen upset U such that $x \in U$ and $y \notin U$.

The functors ClopUp and Spec generalize to a correspondence between Priestley spaces and distributive lattices, yielding Priestley duality

Theorem 2 (Priestley duality)**.** *The category of distributive lattices and lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order-preserving maps.*

[1] All lattices will be assumed to be bounded and lattice homomorphisms to preserve the bounds.

We are now ready to describe the construction dual to taking the free Gödel algebra over a distributive lattice. Let X be a Priestley space. A *chain* (i.e., a totally ordered subset) of X is said to be *closed* when it is closed in the topology on X. We denote by $\mathsf{CC}(X)$ the set of all nonempty closed chains of X. Equip $\mathsf{CC}(X)$ with the Vietoris topology, which is generated by the subbasis $\{\Box V, \Diamond V \mid V \text{ clopen of } X\}$, where

$$\Box V := \{C \in \mathsf{CC}(X) \mid F \subseteq V\} \qquad \text{and} \qquad \Diamond V := \{C \in \mathsf{CC}(X) \mid F \cap V \neq \varnothing\}.$$

Define a partial order $\trianglelefteq$ on $\mathsf{CC}(X)$ by setting $C_1 \trianglelefteq C_2$ iff C_2 is an upset inside C_1.

The following is our main result, characterizing the Esakia duals of free Gödel algebras over distributive lattices.

Theorem 3.

1. *If X is a Priestley space, then $\mathsf{CC}(X)$ is an Esakia root system.*
2. *Let L be a distributive lattice and X its Priestley dual. Then the Gödel algebra dual to $\mathsf{CC}(X)$ is free over L.*

Let $\mathsf{2}$ be the Priestley space consisting of the 2-element chain with the discrete topology. It is well known that for a set S, the ordered topological space $\mathsf{2}^S$ with the product topology and componentwise order is a Priestley space, and that its dual distributive lattice is free over the set S. As a consequence of this observation and Theorem 3, we obtain a dual description of free Gödel algebras.

Corollary 1. *Let S be a set. Then the Gödel algebra dual to $\mathsf{CC}(\mathsf{2}^S)$ is free over the set S.*

While products of Priestley spaces are simply cartesian products, the products in the category of Esakia spaces are difficult to describe. Consequently, computing coproducts of Heyting algebras is a nontrivial task. A generalization of the construction of universal models was employed in [14] to study the finite depth part of the product of two finite Esakia spaces, and the step-by-step method has been employed in [2] to obtain a dual description of binary products of Esakia spaces. We adapt our machinery to describe arbitrary products in the category of Esakia root systems, generalizing the description of binary products of finite Esakia root systems from [7]. As a consequence of Esakia duality, we obtain a dual description of coproducts of any family of Gödel algebras without restrictions on the cardinalities of the family and of its members.

Definition 5. Let $\{Y_i \mid i \in I\}$ be a family of Esakia root systems. Let $\prod_i Y_i$ denote the cartesian product with the componentwise order and product topology, and $\pi_i \colon \prod_i Y_i \to Y_i$ the projection onto the i-th component. We define

$$\bigotimes_{i \in I} Y_i := \{C \in \mathsf{CC}\big(\textstyle\prod_i Y_i\big) \mid \pi_i[C] \text{ is an upset of } Y_i \text{ for every } i \in I\}$$

and equip it with the subspace topology and order induced by $\mathsf{CC}\big(\prod_i Y_i\big)$.

Theorem 4.

1. *Let* $\{Y_i \mid i \in I\}$ *be a family of Esakia root systems. Then* $\bigotimes_{i\in I} Y_i$ *is their product in the category of Esakia root systems.*

2. *Let* $\{G_i \mid i \in I\}$ *be a family of Gödel algebras and* Y_i *their Esakia duals. Then* $\bigotimes_i Y_i$ *is dual to the coproduct of* $\{G_i \mid i \in I\}$ *in* GA.

The proper subvarieties of GA form a countable chain of order type ω, and each of them is axiomatized over GA by a bounded depth axiom [9, 15] (see [17] for the corresponding characterization of the extensions of the Gödel-Dummett logic). We denote by GA_n the subvariety of GA consisting of all the Gödel algebras validating the bounded depth n axiom, and we refer to its members as GA_n-algebras. Replacing $\mathsf{CC}(X)$ with its subspace $\mathsf{CC}_n(X)$, consisting of the nonempty chains in X of size at most n, yields analogues of Theorems 3 and 4 that provide dual descriptions of free GA_n-algebras over distributive lattices and of coproducts in GA_n.

A Heyting algebra is called a *bi-Heyting algebra* if its order dual is also a Heyting algebra. The step-by-step method allows to show that every Heyting algebra free over a finite distributive lattice is a bi-Heyting algebra [12]. Using Theorem 3, we provide a characterization of the Gödel algebras free over distributive lattices that are bi-Heyting algebra. As a consequence, we deduce that free Gödel algebras are always bi-Heyting algebras. Surprisingly, we also show that the situation is very different for free GA_n-algebras.

Theorem 5.

1. *Let* G *be a Gödel algebra free over a distributive lattice* L*. Then* G *is a bi-Heyting algebra iff the order dual of* L *is a Heyting algebra.*

2. *All free Gödel algebras are bi-Heyting algebras.*

3. *A free* GA_n*-algebra is a bi-Heyting algebra iff it is finitely generated, and hence finite.*

These results have been collected in the manuscript [6].

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Preservation Theorems for Many-valued Logics via Categorical Methods

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A canonical result in model theory is the Homomorphism Preservation Theorem (h.p.t.) which states that a first-order formula is preserved under homomorphisms on all structures if and only if it is equivalent to an existential-positive formula. A first order sentence φ in a vocabulary σ is *preserved under homomorphisms* iff, whenever $M \models \varphi$ and there is a homomorphism of σ-structures $M \to N$ then $N \models \varphi$. A sentence ψ is *existential-positive* when it is constructed using only the connectives $\wedge, \vee$ and $\exists$. The h.p.t. is an example of a preservation theorem, a family of results linking a syntactic class of formulas with preservation under a particular kind of map and which are standardly proved via compactness arguments. Rossman [6] established that the h.p.t. remains valid when restricted to finite structures.

Finite Homomorphism Preservation Theorem A first-order sentence of quantifier-rank n is preserved under homomorphisms on finite structures iff it is equivalent in the finite to an existential-positive sentence of quantifier rank $\rho(n)$ (for some explicit function $\rho : \omega \to \omega$).

This is a significant result in the field of finite model theory. It stands in contrast to other results proved via compactness, including the other preservation theorems, where the failure of the compactness also results in the failure of preservation theorem. [5] Indeed, Rossman's proof is a compactness free proof of the finite h.p.t. that can be retroactively carried out in the general case. More than providing a simple alternative proof, by avoiding compactness one maintains control of the syntactic shape of the equivalence existential positive sentence ψ. In particular this allows a comparison of the *quantifier rank* of the original sentence and its existential-positive equivalent (the ρ referenced in the theorem). In the general case this yields the *equirank* h.p.t. where the quantifier rank of the equivalent sentence is the same as original.

Adjacently, Dellunde and Vidal [4] established that a version of the h.p.t. holds for a collection of many-valued models - those defined over a fixed finite MTL-chain. In prior work [3] we showed how one can extend Rossman's proof of a finite h.p.t. to many-valued models defined over the slightly more general UL-chains, in particular establishing a finite variant to Dellunde and Vidal's result. In the many-valued setting both the notion of homomorphism and existential-positive formulas split into a number of interrelated concepts and this naturally provides a number of possible generalisations of the classical h.p.t. Many can be immediately ruled unviable by observing that the 'easy' direction of the theorem fails, i.e. that formulas in the syntactic class are not preserved by the given morphisms. The viable variant directly recoverable from the classical case links homomorphisms with existential-positive sentences (∃.p) understood as they are classically. Namely homormohpsims are maps which preserve the modelling of atomic formula and existential-positive sentences those constructed from just $\wedge, \vee$ and $\exists$ (ignoring any additional algebraic connectives present in the algebraic signature).

Fixed Finite Homomorphism Preservation Theorem Let $\mathscr{P}$ be a predicate language, A UL-chain and φ a consistent $\mathscr{P}$ sentence over A in the finite. Then φ is equivalent over A in the finite to an ∃.p sentence ψ iff φ is preserved under homomorphisms. That is, there is an ∃.p-sentence $\psi : Mod^A_{fin}(\varphi) = Mod^A_{fin}(\psi)$ iff $Mod^A_{fin}(\varphi)$ is closed under homomorphisms.

This generalisation of the classical h.p.t. is one way to precisely express the idea that the 'classical part' of many-valued models still behaves classical. Indeed, the strategy to extend Rossman's result is via a 'classical counterpart' to any given many-valued model, where one demonstrates that this structure behaves well with respect to both homomorphisms and ∃.p-formulas. This motivates an attempt to study potential preservation theorems for many-valued models that directly address their many-valued nature. One viable preservation theorem with a more substantive many-valued character links *monomorphisms* with *strong existential-positive sentences.*

Definition Let σ be a (relational) signature and A a complete lattice. A map $f : M \to N$ between two σ-models is a monomorphism iff for all $R \in \sigma$ and $\bar{m} \in M$ $R^M(\bar{m}) \leqslant R^N(f(\bar{n}))$.

A sentence ψ is said to be strong existential-positive iff it is constructed using only the existential quantifier $\exists$ and algebraic connectives $\circ$ which are order preserving in all arguments with respect to the lattice order.

A promising strategy to pursue such a result comes from recent work by Abramsky and Reggio [2]. Building on previous work on game comonads [1], where various model comparison games are studied through comands on the category of relational structures, they developed an category theoretic framework which is used to prove an abstract homomorphism preservation theorem. The general categories of interest are axiomatized as *arboreal categories* upon which abstracted notions such as game and back-and-forth system are defined. This requires a form of *resource indexing* yielding a family of subcategories for each $k \in \omega$. These transfer to an extensional category via similarly indexed family of adjunctions called a *resource indexed arboreal adjunction* (RIAA). The resulting homomorphism preservation theorem links preservation by the morphisms of the extensional category to the existence of a morphism between the adjoint images in the arboreal category (denoted $M \to^E_k N$). The classical homomorphism preservation theorem, and its finite variant due to Rossman, then emerge as one of the prime examples with the category of relational structures equipped with the Ehrenfeucht-Fraïssé RIAA. To recover the familiar h.p.t. we require an alignment of the morphism existence relation which in turn is tied to the existence of a single equivalent formula. More explicitly, for a set of models $\mathscr{D}$ we say that $\mathscr{D}$ is upwards closed with respect to a relation ∇ iff $\forall M, N \in \mathscr{D}$ if $M \in \mathscr{D}$ and $M \nabla N$ then $N \in \mathscr{D}$. The two critical lemmas are then:

Lemma 1 For all σ-structures M, N and all $k > 0$ we have $E_k(M) \to E_k(N)$ iff $M \Rightarrow^{\exists^+ FO_k} N$.

Lemma 2 For all $k \geqslant 0$ and all full subcategory $\mathscr{D}$ of Struct(σ) $\mathscr{D} = Mod(\psi)$ for some $\exists^+ FO_k$ iff $\mathscr{D}$ is upwards closed with respect to the relation $\Rightarrow^{\exists^+ FO_k}$, defined as for all $\psi \in \exists^+ FO_k$ $M \models \psi$ implies $N \models \psi$.

They further establish a sufficiency condition for the HP to for a given RIAA, namely the satisfaction of a series of axioms, two concerning the extensional category (E1-E2) and four concerning the RIAA adjunction itself (A1-A4). This is used in particular to establish the abstract HP for the Ehrenfeucht-Fraïssé RIAA.

In this talk we outline the attempt to adapt and apply this framework to many-valued models. This begins with an outlining of the basic behaviour of many-valued models under monomorphisms. In the classical case this category is defined relative to a propositional signature, in the many-valued case it is also defined relative to a fixed complete lattice A.

Proposition Let A be a complete lattice. We use $\mathscr{M}(\mathscr{P})$ to denote the category whose objects are $\mathscr{P}$-models defined over A. The category $\mathscr{M}(\mathscr{P})$ is complete and co-complete.

The next step is the construction of a suitably adjusted arboreal category and RIAA linking it to the category $\mathscr{M}(\mathscr{P})$.

These constructions establish that one can fit the category of many-valued models into the framework of Abramsky and Reggio. The obvious route to then establish a concrete preservation theorem for monomorphisms is to establish the statements E1-E2 and A1-A4 hold, and the two lemmas transferring the abstract HP result to the familiar one. At the time of writing these remain conjectural. As is often the case when working with many-valued models the answer is sensitive to the behaviour of the underlying algebra. Initial work suggests that when A is finite adaptions of the classical Ehrenfeucht-Fraïssé RIAA example will suffice, but when A is infinite the situation is significantly more complex.

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Strong completeness for the predicate logic of the continuous t-norms

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Abstract

The axiomatic system introduced by Hájek axiomatizes first-order logic based on BL-chains. In this study, we extend this system with the axiom $(\forall x\varphi)^2 \leftrightarrow \forall x\varphi^2$ and the infinitary rule

$$\frac{\varphi \vee (\alpha \to \beta^n) : n \in \mathbb{N}}{\varphi \vee (\alpha \to \alpha \& \beta)}$$

to achieve strong completeness with respect to continuous t-norms.

In [8], the author proposed the study of first-order many-valued logics interpreting universal and existential quantifiers as infimum and supremum, respectively, on a set of truth values. From the 1963 article by [5], it follows that the infinitary rule

$$\frac{\varphi \oplus \varphi^n : n \in \mathbb{N}}{\varphi}$$

can be added to the first-order Łukasiewicz calculus to obtain weak completeness with respect to the Łukasiewicz t-norm. [6] later axiomatized first-order Gödel logic in 1969.

[4] provided a general approach to first-order fuzzy logic, introducing a syntactic logic, denoted by BL∀, which is strongly complete with respect to models based on BL-chains. However, the problem of finding an appropriate syntactic logic for models based on continuous t-norms remained unresolved.

In the propositional case, [4] exhibited a syntactic logic that is strongly complete with respect to valuations on BL-chains. [7] later proved that by adding the infinitary rule

$$\frac{\varphi \vee (\alpha \to \beta^n) : n \in \mathbb{N}}{\varphi \vee (\alpha \to \alpha \& \beta)}$$

to the syntactic logic, a strong completeness result can be achieved with respect to valuations on t-norms.

In the first-order case, quantifiers can exhibit distinct behaviors in a continuous t-norm compared to a generic BL-chain. For example, the sentence,

$$\forall x(\varphi \& \varphi) \to ((\forall x \varphi) \& (\forall x \varphi)) \tag{RC}$$

is true in models based on continuous t-norms and is not true in general. Moreover, [3] demonstrated that standard first-order tautologies coincide with first-order tautologies over complete BL-chains satisfying (RC).

In this paper we show that by adding Kułacka's Infinitary Rule and Hájek and Montagna's axiom RC to BL∀, a strong standard completeness result can be proven for models based on continuous t-norms. Thus, this paper aims to contribute to the study of first-order extensions of propositional logics, such as [1], [2], and [3].

We introduce our logic extending the logic BL∀ with an additional axiom and an infinitary rule and utilize a Henkin construction to demonstrate that for a given theory Γ and a sentence φ such that $\Gamma \nvdash \varphi$, there exists an expanded theory Γ^* that also satisfies $\Gamma^* \nvdash \varphi$ and possesses additional properties (Henkin property and *prelinearity*) necessary for the subsequent construction of a desirable Lindenbaum algebra. Then, a Lindenbaum algebra is constructed and embedded in a continuous t-norm, with the help of a new version of a weak saturation result, providing the final prerequisite for the strong completeness theorem.

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Degree of Kripke Incompleteness in Tense Logics

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Introduction

Kripke-completeness of modal logics has been extensively studied since 1960s. Thomason [13] established the existence of Kripke-incomplete tense logics, that is, tense logics which are not complete with respect to any class of Kripke frames. Later, Fine [8] and van Benthem [14] found examples of Kripke-incomplete modal logics. Fine [8] raised a question concerning the degree of Kripke-incompleteness of logics in the lattice $\mathsf{NExt}(\mathsf{K})$ of all normal modal logics. In general, for each lattice $\mathscr{L}$ of logics and $L \in \mathscr{L}$, the *degree of Kripke-incompleteness* $\mathsf{deg}_{\mathscr{L}}(L)$ *of* L *in* $\mathscr{L}$ is defined as:

$$\mathsf{deg}_{\mathscr{L}}(L) = |\{L' \in \mathscr{L} : \mathsf{Fr}(L') = \mathsf{Fr}(L)\}|.$$ [1]

In other words, the degree of Kripke-incompleteness of L in $\mathscr{L}$ is the cardinality of logics in $\mathscr{L}$ which share the same class of Kripke-frames with L. A logic L is *strictly Kripke-complete in* $\mathscr{L}$ if $\mathsf{deg}_{\mathscr{L}}(L) = 1$. A celebrated result on Kripke-incompleteness is the dichotomy theorem for degree of Kripke-incompleteness in $\mathsf{NExt}(\mathsf{K})$ by Blok [3]: every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is of the degree of Kripke-incompleteness 1 or $2^{\aleph_0}$. This theorem was proved in [3] algebraically by showing that union splittings in $\mathsf{NExt}(\mathsf{K})$ are exactly the consistent strictly Kripke-complete logics and all other consistent logics have the degree $2^{\aleph_0}$. A proof based on Kripke semantics was given later in [4]. This characterization of the degree of Kripke-incompleteness indicates locations of Kripke-complete logics in the lattice $\mathsf{NExt}(\mathsf{K})$.

Further results have been obtained on generalizations of degree of Kripke-incompleteness. The degree of modal incompleteness with respect to neighborhood semantics was investigated in [7, 10, 5]. Dziobiak [7] proved the dichotomy theorem for degree of incompleteness in the lattice $\mathsf{NExt}(\mathsf{D} \oplus (\Box^n p \to \Box^{n+1} p))$ w.r.t neighborhood semantics for all $n \in \omega$. Litak [10] studied modal incompleteness w.r.t Boolean algebras with operators (BAOs) and showed the existence of a continuum of neighborhood-incomplete modal logics extending Grz. For more on modal incompleteness from an algebraic view, we refer the readers to [11]. Degree of finite model property (FMP) was introduced in [1], where the following

[1] To simplify notation, we always write deg_{L_0} for $\mathsf{deg}_{\mathsf{NExt}(L_0)}$.

anti-dichotomy theorem for the degree of FMP for extensions of the intuitionistic propositional logic IPC was proved: for each cardinal κ with $0 < \kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{Ext}(\mathsf{IPC})$ such that the degree of FMP of L in $\mathsf{Ext}(\mathsf{IPC})$ is κ. It was also shown in [1] that the anti-dichotomy theorem of the degree of FMP holds for $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. Degrees of FMP in bi-intuitionistic logics were studied in [6].

It is a longstanding open problem whether Blok's dichotomy theorem holds for extensions of transitive modal logics such as K4 and S4, or for extensions of the intuitionistic logic IPC. Since the Blok's proof relies on non-transitive frames heavily, we need new technique to solve these problems.

Tense logics are bi-modal logics that include a future-looking necessity modality $\Box$ and a past-looking possibility modality $\blacksquare$, of which the lattices are substantially different from those of modal logics (see [9, 13, 12]). However, as far as we know, no characterization of the degree of Kripke-incompleteness in lattices of tense logics is known. In this work, we study Kripke-incompleteness in tense logics. We start with the lattice $\mathsf{NExt}(\mathsf{K4}_t)$ of transitive tense logics. Inspired by the proof for Blok's dichotomy theorem in [4], we prove the dichotomy theorem for transitive tense logics, that is, every tense logic $L \in \mathsf{NExt}(\mathsf{K4}_t)$ is of degree of Kripke-incompleteness 1 or $2^{\aleph_0}$. By a similar argument, we also show that dichotomy theorem of the degree of Kripke-incompleteness holds for $\mathsf{NExt}(\mathsf{K}_t)$.

Main Results

Let $L^* = \mathsf{K4}_t \oplus (\Diamond\top \vee \blacklozenge\top)$ and $L^\circ = \mathsf{K}_t \oplus (\Diamond\top \vee \blacklozenge\top)$. Our main result is the following theorem:

Theorem 1. *Let $L \in \mathsf{NExt}(\mathsf{K}_t)$. Then the following holds:*

1. *If $L \in \{\mathsf{K}_t, L^\circ\}$, then $\mathsf{deg}_{\mathsf{K}_t}(L) = 1$. Otherwise $\mathsf{deg}_{\mathsf{K}_t}(L) = 2^{\aleph_0}$.*
2. *Suppose $L \in \mathsf{NExt}(\mathsf{K4}_t)$. If $L \in \{\mathsf{K4}_t, L^*\}$, then $\mathsf{deg}_{\mathsf{K4}_t}(L) = 1$. Otherwise $\mathsf{deg}_{\mathsf{K4}_t}(L) = 2^{\aleph_0}$.*

Dichotomy theorems for tense logics and transitive tense logics follow from Theorem 1. An interesting corollary is that even the inconsistent tense logic $\mathcal{L}_t$ is of degree of Kripke-incompleteness $2^{\aleph_0}$, which means that there are continuum many logics in $\mathsf{NExt}(\mathsf{K4}_t)$ with no Kripke frame.

Proof Idea

In what follows, we report on the proof idea of Theorem 1(2) and the main technique used.

Definition 1. A *Kripke frame* is a pair $\perp = (X, R)$ where $X \neq \varnothing$ and $R \subseteq X \times X$. The *inverse* of R is defined as $\breve{R} = \{\langle v, w\rangle : wRv\}$. For every $w \in X$, let $R[w] = \{u \in X : wRu\}$ and $\breve{R}[w] = \{u \in X : uRw\}$. For every $U \subseteq W$, we define $R[U] = \bigcup_{x\in U} R[x]$ and $\breve{R}[U] = \bigcup_{x\in U} \breve{R}[x]$.

For $k \geqslant 0$, we define $R^k_\sharp[w]$ by $R^0_\sharp[w] = \{w\}$ and $R^{k+1}_\sharp[w] = R^k_\sharp[w] \cup R[R^k_\sharp[w]] \cup \breve{R}[R^k_\sharp[w]]$. Let $R^\omega_\sharp[w] = \bigcup_{k\geqslant 0} R^k_\sharp[w]$. For all binary relation R, we write R^+ for its transitive closure.

Intuitively, $R^n_\sharp[w]$ is the set of all points which can be reached from w by an $(R \cup \breve{R})$-path of length no more than n. Models, truth and validity of tense formulas are defined as usual. For each $n \in \omega$ and $\varphi \in \mathcal{L}_t$, we define the formula $\Delta^n\varphi$ by: $\Delta^0\varphi = \varphi$ and $\Delta^{k+1}\varphi = \Delta^k\varphi \vee \Diamond\Delta^k\varphi \vee \blacklozenge\Delta^k\varphi$. Then the readers can verify that $\widehat{K}, w \models \Delta^n\varphi$ iff $\widehat{K}, u \models \varphi$ for some $u \in R^n_\sharp[w]$.

Lemma 1. *Let $L \in \mathsf{NExt}(\mathsf{K4}_t)$. Then $L \in \{\mathsf{K4}_t, L^*\}$ implies $\mathsf{deg}_{\mathsf{K4}_t}(L) = 1$.*

Proof. By Kripke-completeness of $\mathsf{K4}_t$, $\mathsf{deg}_{\mathsf{K4}_t}(\mathsf{K4}_t) = 1$. To show $\mathsf{deg}_{\mathsf{K4}_t}(L^*) = 1$, suppose there exists $L' \in \mathsf{NExt}(\mathsf{K4}_t)$ such that $\mathsf{Fr}(L') = \mathsf{Fr}(L^*)$ and $L' \neq L^*$. Since L^* is Kripke-complete, $L' \subsetneq L^*$ and so $\Diamond\top \vee \blacklozenge\top \notin L'$. Thus $(\{0\}, \varnothing) \in \mathsf{Fr}(L')$, which contradicts to $\mathsf{Fr}(L') = \mathsf{Fr}(L^*)$. □

To prove the second half of Theorem 1(2), we need some auxiliary frame-constructions which can bring us frames containing long enough zigzags. *In what follows, by frames we mean rooted transitive frames.* Let us recall the book-construction of frames from [9, Section 3]. Consider frames $\perp = (X, R)$ and $\mathfrak{G} = (Y, S)$ such that $X \cap Y = \{u\}$. Then $\mathfrak{H} = ((X \cup Y), (R \cup S)^+)$ is a frame such that $\mathfrak{H}{\upharpoonright}X \cong \perp$ and $\mathfrak{H}{\upharpoonright}Y \cong \mathfrak{G}$. Since we can always re-label points in domains of frames, by similar idea, for all frames $\perp = (X, R)$, $\mathfrak{G} = (Y, S)$ and points $w \in X$ and $u \in Y$, we can construct the *combination* $\langle \perp w{+}u\mathfrak{G}\rangle$ *of* $(\perp, w)$ *and* $(\mathfrak{G}, u)$ by 'gluing' $\perp$ and $\mathfrak{G}$ at w and u.

Let $\perp = (X, R)$ be a frame. For any $n \in \mathbb{Z}^+$ and fixed points $w, u \in X$, the n-pages book $\perp^n_{w,u}$ of $\perp$ is constructed by combining n copies of $\perp$ at w and u alternatively. An example of the book construction is given in Figure 1. It is not hard to verify that $\perp$ is a t-morphic image of $\perp^n_{w,u}$ for each $n \in \mathbb{Z}^+$. As a corollary, for all $x \in X$ and $n \in \mathbb{Z}^+$, $\mathsf{Th}(\perp^n_{w,u}, x) \subseteq \mathsf{Th}(\perp, x)$.[2] Moreover, if $\langle w, u\rangle \in R \setminus \breve{R}$, then the book-construction can provide us frames with long enough zigzags. Formally, the following lemma holds:

Lemma 2. *Let $\perp = (X, R)$, $w, u \in X$, Rwu and $u \notin R[w]$. Let $n \in \omega$ and $\mathfrak{G} = (Y, S) = \perp^{4n+2}_{w,u}$. Then $S^n_\sharp[x] \neq Y$ holds for all $x \in X$.*

Now we start to prove the second half of Theorem 1(2). Let $L \in \mathsf{NExt}(\mathsf{K4}_t)$ be an arbitrarily fixed logic such that $L \notin \{\mathsf{K4}_t, L^*\}$. Then $L \nsubseteq L^*$. Take any $\varphi_L \in L \setminus L^*$. Then we can show that φ_L is refuted by some finite non-symmetric frame. By Lemma 2, we can prove the following lemma:

[2] For all frames $\perp = (X, R)$ and $x \in X$, we define $\mathsf{Th}(\perp, x) = \{\varphi \in \mathcal{L}_t : \perp, x \models \varphi\}$.

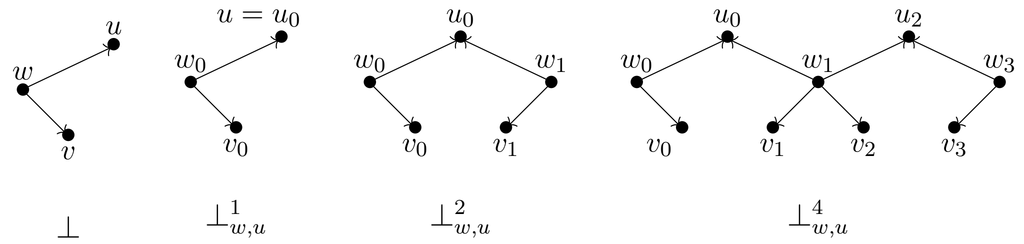

Figure 1: Examples for the book construction

Lemma 3. *There is a finite frame* $\bot_L$ *and* $w_L, u_L \in X$ *such that* $\bot_L, w_L \not\models \varphi_L$ *and* $u_L \notin R_\sharp^{\mathsf{md}(\varphi)}[w_L]$.

Let $\mathbb{Z}^\flat = \omega \setminus \{0, 1\}$. For each $I \in \mathscr{P}(\mathbb{Z}^\flat)$, we define the general frame $\mathbb{F}_I = (X_I, R_I, P_I)$ as follows:

- $X_I = X_L \uplus (\omega \cup \{i^* : i \in I\})$.
- $R_I = (R_L \cup \{\langle n, m\rangle \in \omega \times \omega : n < m\} \cup \{\langle i^*, i\rangle : i \in I\} \cup \{\langle 0, u_L\rangle\})^+$.
- P_I is the tense algebra generated by $\mathscr{P}(X_L)$.

Example 1. Let $\mathbb{P}$ be the set of all prime numbers. Then $\bot_\mathbb{P}$ is depicted by Figure 2.

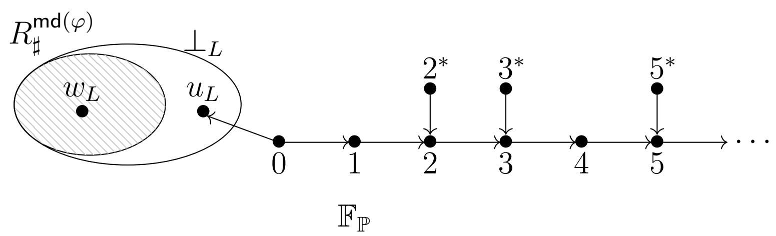

Figure 2: The frame $\mathbb{F}_\mathbb{P}$

Let $I \in \mathscr{P}(\mathbb{Z}^\flat)$ be arbitrarily fixed and take the minimal $k \in \omega$ such that $|\bot_L| < k$ and $X_I = R_{I\sharp}^{k}[v]$ for all $v \in X_I$. For each $n \in \omega$ and $m \in \mathbb{Z}^\flat$, we define the formulas γ_n and γ_m^* as follows:

- $\gamma_0 = \blacksquare\bot \wedge \Diamond\blacksquare^2\bot \wedge \Diamond^k\blacksquare^{k+1}\bot$ and $\gamma_{l+1} = \blacklozenge\gamma_l \wedge \blacksquare^2\neg\gamma_l$.
- $\gamma_m^* = \Diamond\gamma_m \wedge \Box\neg\gamma_{m-1} \wedge \blacksquare\bot$.

Then we can verify that for all $n \in \omega$ and $m \in I$, the constant formulas γ_n and γ_m^* are true at exactly points n and m^*, respectively. Let $L_I = \mathsf{Log}(\mathsf{Fr}(L) \cup \{\mathbb{F}_I\})$. Clearly, $L_I \subseteq \mathsf{Log}(\mathsf{Fr}(L))$ and so $\mathsf{Fr}(L) = \mathsf{Fr}(\mathsf{Log}(\mathsf{Fr}(L))) \subseteq \mathsf{Fr}(L_I)$. Note that for all distinct $I, J \in \mathscr{P}(\mathbb{Z}^\flat)$, $L_I \neq L_J$. Indeed, take any $I \not\subseteq J$ and $i \in I \setminus J$, we can show that $\neg\varphi_L \to \Delta^k\gamma_i^* \in L_I \setminus L_J$. Moreover, we have

Lemma 4. $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ *for all* $I \in \mathbb{Z}^\flat$.

Proof. (Sketch.) Suppose $\mathsf{Fr}(L) \neq \mathsf{Fr}(L_I)$. Then $\mathsf{Fr}(L) \subsetneq \mathsf{Fr}(L_I)$ and there is a frame $\mathfrak{G} = (Y, S) \in \mathsf{Fr}(L_I)$ and $y \in Y$ such that $\mathfrak{G}, y \not\models \psi$ for some $\psi \in L$. Thus $\mathbb{F}_I \models \Delta^k(\gamma_0 \wedge \Diamond\gamma_1)$ and so $\neg\psi \to \Delta^k(\gamma_0 \wedge \Diamond\gamma_1) \in L_I$. Since $\psi \in L$, $\mathbb{F}_I, 0 \models \Box(\Box p \to p) \to \Box p$ and $\mathbb{F}_I, 0 \models \Box(\gamma_i \to \Diamond\gamma_{i+1})$ for all $i \in \omega$, we have

$$\Delta^k\neg\psi \wedge \gamma_0 \to (\Box(\Box p \to p) \to \Box p) \in L_I \text{ and } \{\Delta^k\neg\psi \wedge \gamma_0 \to \Box(\gamma_i \to \Diamond\gamma_{i+1}) : i \in \omega\} \subseteq L_I.$$

Since $\mathfrak{G} \not\models \psi$ and $\mathfrak{G} \models \neg(\gamma_i \leftrightarrow \gamma_j)$ for any different $i, j \in \omega$, there exists an infinite strict S-chain $\langle u_i : i \in \omega \rangle \subseteq U$ such that $y = u_0$ and $\mathfrak{G}, u_i \models \gamma_i$ for all $i \in \omega$. Take any propositional variable $p \in \mathsf{Prop}$ which does not occur in ψ. Then we see that $\mathfrak{G}, u_0 \not\models \Box(\Box p \to p) \to \Box p$. Hence $\mathfrak{G} \not\models \Delta^k\neg\psi \wedge \gamma_0 \to (\Box(\Box p \to p) \to \Box p)$, which contradicts $\mathfrak{G} \models L_I$. $\Box$

Since $I \in \mathscr{P}(\mathbb{Z}^\flat)$ is arbitrarily fixed and $|\mathscr{P}(\mathbb{Z}^\flat)| = 2^{\aleph_0}$, we conclude that $\mathsf{deg}_{\mathsf{K4}_t}(L) = 2^{\aleph_0}$. Note that $L \notin \{\mathsf{K4}_t, L^*\}$ is also chosen arbitrarily, the proof of Theorem 1(2) is concluded.

To show Theorem 1(1), take any $L \notin \{\mathsf{K}_t, L^\circ\}$. Then $L \nsubseteq L^\circ$ and there exists $\varphi_L \in L \setminus L^\circ$. By the non-transitive book-construction in [9] or the unrevealing construction introduced in [2], Lemma 3 holds. For each $I \in \mathscr{P}(\mathbb{Z}^\flat)$, we define the general frame $\mathbb{F}'_I = (X_I, R'_I, P_I)$, where $R_I = R_L \cup \{\langle n, m\rangle : n < m\} \cup \{\langle i^*, j\rangle : i \in I \text{ and } i \leqslant j\} \cup \{\langle 0, u_L\rangle\}$. By similar arguments, $L'_I = \mathsf{Log}(\mathsf{Fr}(L) \cup \{\mathbb{F}'_I\})$ share the same frames with L and $|\{L'_I : I \subseteq \mathbb{Z}^\flat\}| = 2^{\aleph_0}$.

In fact, Theorem 1 is also a generalization of Blok's characterization of the degree of Kripke-incompleteness of modal logics. It follows from [9, Theorem 22] that $\{\mathsf{K}_t, L^\circ\}$ and $\{\mathsf{K4}_t, L^*\}$ are the sets of union splittings in $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$, respectively. Thus we have

Theorem 2. *Let* $L_0 \in \{\mathsf{K}_t, \mathsf{K4}_t\}$ *and* $L \in \mathsf{NExt}(L_0)$ *be consistent. If L is a union splitting in* $\mathsf{NExt}(L_0)$*, then* $\mathsf{deg}_{L_0}(L) = 1$. *Otherwise* $\mathsf{deg}_{L_0}(L) = 2^{\aleph_0}$.

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Two-Layered Modal Logics: A New Beginning

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This talk presents a new approach logic with two-layered modal syntax. The syntax of these logics is given by:

- *inner* formulas build from inner variables using given *inner* propositional connectives
- *atomic outer* formulas built from inner formulas using given *modalities*
- *complex outer* formulas are built from the atomic ones using given *outer* propositional connectives

Early examples of such logics were logics of uncertainty based on Hamblin's original idea of reading the atomic outer formulas $P\varphi$ as 'probably φ' [22] and semantically interpreting it (in a given Kripke frame equipped with a probability measure) as *true* iff the probability of the set of worlds where φ is true is bigger than a given threshold. This idea was later elaborated and extended by Fagin, Halpern and many others; see e.g. [9, 21].

These initial examples used classical logic to govern the behavior of formulas on both layers. A departure from this paradigm was proposed by Hájek and Harmancová in [20] which they later developed in collaboration with Godo and Esteva in [19]. They kept classical logic to govern the inner layer of events, but proposed Łukasiewicz logic to govern the outer layer of statements on probabilities of these events. The truth degree of the atomic outer formula $P\varphi$ could then be directly identified with the probability of the set of worlds where φ is true. Later, other authors changed even the logic governing the inner layer (e.g., another fuzzy logic in order to allow for the treatment of uncertainty of vague events) or considered additional (possibly non-unary) modalities (e.g. for conditional probability), see e.g. [17, 16, 25, 18, 12, 11, 8, 14].

This research thus gave rise to an interesting way of combining logics which allows to use one logic to reason about formulas (or rules) of another one with numerous examples described and developed in the literature. In our previous work [7], we took the first steps towards development of a general theory of such logics and proved, in a rather general setting, two forms of completeness theorem most commonly appearing in the literature. Although the level of generality seemed quite sufficient back then (*finitary weakly implicative logics with unit and lattice conjunction*, see [4]), recent developments in the field show the need for more: e.g., the inner logic in [2] and the outer logic in [1] are not weakly implicative, and in the former case they are not even equivalential.

In an LATD 2022 talk we presented an expansion our theory of [7] to cover even those examples, the resulting formalism was however rather cumbersome. In this talk, we propose a radical departure from the usual paradigm by taking, as elementary, the **consequence relation between equations rather than formulas.** In many situations it is just a notational variant, but in all cases it dramatically simplifies and clarifies the used formalism and the proofs on the main results. Our second contribution is a new proof of the completeness result for finitary logics which has usually involved a rather complex and cumbersome syntactical translation. In our approach, we first prove the completeness result of the related infinitary logics (for which we need no translations) and then easily transform it into the desired result for the finitary case.

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Intuitionistic monotone modal logic

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Introduction

Ever since Fitch first introduced a modal extension of intuitionistic logic [7], many different such logics have been put forward. These often differ in their connection between the modal operators, ranging from no connection at all [2, 16] to one modality being definable from the other [3], [2, Section 11]. The logic IK is one of myriad intuitionistic modal logics with a more subtle interaction between the modalities [6, 13, 5, 14]. It can be obtain by embedding modal logic into first order logic, and then changing this first order logic to an intuitionistic one.

Akin to the normal modal case, there are various intuitionistic modal logics with monotone modalities [9, 15, 1, 4]. Interestingly, none of these is obtained from classical monotone modal logic [11] in the same way IK arises from normal modal logic. We close this gap by defining the intuitionistic modal logic $\mathsf{IM_s}$[1] as the set of formulas whose standard translation is derivable in a suitable intuitionistic first-order logic, and axiomatising the resulting logic.

Throughout this abstract, we denote by $\mathscr{L}$ the language generated by the grammar

$$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where p ranges over some set Prop of proposition letters. This can be viewed as the language underlying classical and intuitionistic modal logic with normal or monotone modalities. The first step in our quest for $\mathsf{IM_s}$ is to define a suitable first-order logic FOM that describes (classical) monotone modal logic.

Going monotone

Monotone modal logic M is the extension of classical propositional logic with a modal operator $\Box$ that satisfies $(p \to q)/(\Box p \to \Box q)$. We can define $\Diamond = \neg\Box\neg$, which will be monotone as well. This logic can be interpreted in so-called neighbourhood models.

Definition 1. A *neighbourhood model* is a tuple (W, γ, V) consisting of a nonempty set W, a *neighbourhood function* $\gamma : W \to \mathscr{P}\mathscr{P}W$, an a valuation $V : \mathrm{Prop} \to \mathscr{P}W$. The

[1] The analogy of our logic with IK suggests naming it IM, but this is already used in [3, 4] for different logics. To avoid confusion, we adopt the name $\mathsf{IM_s}$. The "s" indicates that our logic is stronger than IM from [4].

modal operators can be interpreted via:

$$\begin{array}{lll} \mathfrak{M}, w \Vdash \Box\varphi & \text{iff} & \text{there exists } a \in \gamma(w) \text{ such that for all } v \in a,\ \mathfrak{M}, v \Vdash \varphi \\ \mathfrak{M}, w \Vdash \Diamond\varphi & \text{iff} & \text{for all } a \in \gamma(w) \text{ there exists } v \in a \text{ such that } \mathfrak{M}, v \Vdash \varphi \end{array}$$

Taking a first-order perspective of neighbourhood models is not as easy as for Kripke models, because γ relates worlds to sets of worlds. To get around this, we use a two-sorted language, with one sort representing the worlds and the other the neighbourhoods [8, 12, 10].

Definition 2. Let FOM be the two-sorted first-order logic with sorts world and nbhd, a predicate N between world and nbhd, a predicate E between nbhd and world, and a unary predicate P_i of type world for each $p_i \in \mathrm{Prop}$.

Then a FOM-structure is a tuple $\mathfrak{M} = (D_{\mathsf{w}}, D_{\mathsf{n}}, N, E, P_i)$ consisting of sets D_{w} and D_{n} interpreting the sorts, relations $N \subseteq D_{\mathsf{w}} \times D_{\mathsf{n}}$ and $E \subseteq D_{\mathsf{n}} \times D_{\mathsf{w}}$, and subsets $P_i \subseteq D_{\mathsf{w}}$ for each $p_i \in \mathrm{Prop}$. Every neighbourhood model $\mathfrak{M} = (W, \gamma, V)$ gives rise to a FOM-structure

$$\mathfrak{M}^{\circ} = \Big(W, \bigcup\{\gamma(w) \mid w \in W\}, R_{\gamma}, R_{\ni}, \{V(p_i) \mid p_i \in \mathrm{Prop}\}\Big),$$

where $wR_{\gamma}a$ iff $a \in N(w)$ and $aR_{\ni}w$ iff $w \in a$. In the converse direction, a FOM-structure $\mathfrak{M} = (D_{\mathsf{w}}, D_{\mathsf{n}}, N, E, P_i)$ yields a neighbourhood model $\mathfrak{M}^{\bullet} = (D_{\mathsf{w}}, \gamma, V)$, where $V(p_i) = P_i$ and

$$\gamma(x) = \{\{y \in D_{\mathsf{w}} \mid aEy\} \mid xNa\}.$$

We can define the standard translation $\mathrm{st}_x : \mathscr{L} \to \mathsf{FOM}$ by:

$$\begin{array}{rlrl} \mathrm{st}_x(\top) &= (x = x) & \mathrm{st}_x(\varphi \wedge \psi) &= \mathrm{st}_x(\varphi) \wedge \mathrm{st}_x(\psi) \\ \mathrm{st}_x(\bot) &= (x \neq x) & \mathrm{st}_x(\varphi \vee \psi) &= \mathrm{st}_x(\varphi) \vee \mathrm{st}_x(\psi) \\ \mathrm{st}_x(p_i) &= \mathsf{P}_i x & \mathrm{st}_x(\varphi \to \psi) &= \mathrm{st}_x(\varphi) \to \mathrm{st}_x(\psi) \\ \mathrm{st}_x(\Box\varphi) &= \exists a.x\mathsf{N}a \wedge \forall y.a\mathsf{E}y \to \mathrm{st}_y(\varphi) & \mathrm{st}_x(\Diamond\varphi) &= \forall a.x\mathsf{N}a \to \exists y.a\mathsf{E}y \wedge \mathrm{st}_y(\varphi) \end{array}$$

Proposition 1. *For all $\varphi \in \mathscr{L}$, neighbourhood models $\mathfrak{M}$ and FOM-structures $\mathfrak{N}$, we have:*

$$\mathfrak{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{M}^{\circ} \models \mathrm{st}_x(\varphi)[w], \qquad \mathfrak{N}^{\bullet}, w \Vdash \varphi \quad \textit{iff} \quad \mathfrak{N} \models \mathrm{st}_x(\varphi)[w].$$

Going intuitionistic

We now change the first-order logic FOM to the intuitionistic first-order logic IFOM of the same signature. Note that st_x can be viewed as a translation $\mathrm{st}_x : \mathscr{L} \to \mathsf{IFOM}$. Then we can define:

Definition 3. $\mathsf{IM_s} := \{\varphi \in \mathscr{L} \mid \mathsf{IFOM} \models \mathrm{st}_x(\varphi)\}$.

We can now ask how to axiomatise $\mathsf{IM_s}$. It turns out that we can do so as follows:

Definition 4. Let $\mathsf{IM}_{\mathsf{Ax}}$ be the smallest set of $\mathscr{L}$-formulas that contains an axiomatisation for intuitionistic logic as well as the axioms

$$\Box(p \wedge q) \to \Box p, \qquad \Diamond(p \wedge q) \to \Diamond p, \qquad (\Box p \wedge \Diamond\neg p) \to \bot \quad \text{and} \quad (\Box\top \to \Diamond p) \to \Diamond p,$$

and that is closed under modus ponens, uniform substitution, and the congruence rules

$$\frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q} \quad \text{and} \quad \frac{p \leftrightarrow q}{\Diamond p \leftrightarrow \Diamond q}.$$

We write $\mathsf{IM}_{\mathsf{Ax}} \vdash \varphi$, and say that φ is *derivable*, if φ is in $\mathsf{IM}_{\mathsf{Ax}}$.

Alternatively, we can replace the monotonicity axioms and congruence rules for the *monotonicity rules* $(p \to q)/(\Box p \to \Box q)$ and $(p \to q)/(\Diamond p \to \Diamond q)$ to obtain the same logic.

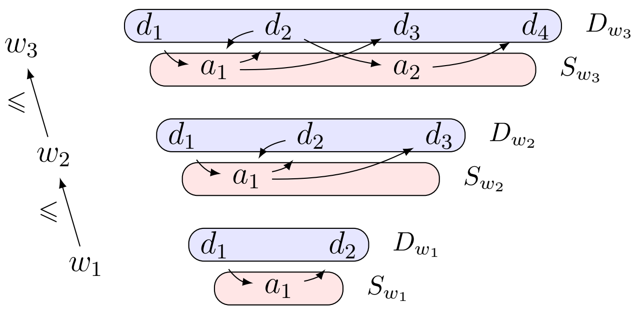
Our main theorem reads:

Theorem 1. *For any $\varphi \in \mathscr{L}$, we have $\varphi \in \mathsf{IM}_{\mathsf{s}}$ if and only if $\mathsf{IM}_{\mathsf{Ax}} \vdash \varphi$.*

Going semantic

One way to prove Theorem 3 is via a semantic detour. This not only allows us to use a routine canonical model construction, but also exposes a new way of thinking about neighbourhood models in an intuitionistic setting. Theorem 3 follows from the following three steps.

Step 1. Use* IFOM*-structures as a semantics for* IM_{s}*. Since IM_{s} is defined via the first-order logic IFOM, we can use the first-order structures for IFOM as a semantics for IM_{s}. An IFOM-structure consists of a poset $(W, \leqslant)$ with at each world $w \in W$ a classical first-order structure, such that these classical structures increase along $\leqslant$. For example:



We can then interpret $\mathscr{L}$-formulas at pairs $\langle w, d\rangle$ where $d \in D_w$. We let $\langle w, d\rangle \Vdash \varphi$ if $w \models \mathrm{st}_x(\varphi)[d]$. Concretely, we let $\langle w, d\rangle \Vdash p_i$ if d is in the interpretation of P_i at w, intuitionistic connectives are interpreted as usual in first-order intuitionistic logic, and:

$\langle w, d\rangle \Vdash \Box\varphi$ iff there exists $a \in A_w$ such that $dN_w a$ and
for all $w' \geqslant w$ and $d' \in D_{w'}$, $aE_{w'}d'$ implies $\langle w, d'\rangle \Vdash \varphi$

$(w, x) \Vdash \Diamond\varphi$ iff for all $w' \geqslant w$ and all $a' \in D_n(w')$,
if $xN_{w'}a'$ then there exists $y' \in D_s(w')$ such that $a'E_{w'}y'$ and $(w', y') \Vdash \varphi$

Let us write $\mathsf{FOS} \Vdash \varphi$ if φ is true in all worlds of all such first-order structures. Then by definition of $\mathsf{IM_s}$ and IFOM we have:

Proposition 2. *For all $\varphi \in \mathscr{L}$, we have $\varphi \in \mathsf{IM_s}$ if and only if $\mathsf{FOS} \Vdash \varphi$.*

Step 2. Define intuitionistic neighbourhood models. Taking inspiration from the first-order semantics, we see that an intuitionistic version of a neighbourhood can change when moving along the intuitionistic accessibility relation. Guided by this, we define an *intuitionistic neighbourhood* as a partial function

$$a : W \rightharpoonup \mathscr{P}W$$

such that $\mathrm{dom}(a) = \{w \in W \mid a(w) \text{ is defined}\}$ is upward closed in $(W, \leqslant)$. An *intuitionistic neighbourhood model* is then given by a tuple $(W, \leqslant, N, V)$ such that $(W, \leqslant)$ is a poset, V is a valuation that assigns to each proposition letter an upset of $(W, \leqslant)$, and N is a collection of intuitionistic neighbourhoods. The modalities are then interpreted as:

$$\begin{array}{lll} \mathfrak{M}, w \Vdash \Box\varphi & \text{iff} & \text{there exists } a \in N \text{ such that } w \in \mathrm{dom}(a) \text{ and} \\ & & \text{for all } w' \geqslant w,\ v \in a(w') \text{ implies } v \Vdash \varphi \\ \mathfrak{M}, w \Vdash \Diamond\varphi & \text{iff} & \text{for all } w' \geqslant w \text{ and all } a \in N \text{ such that } w' \in \mathrm{dom}(a) \\ & & \text{there exists } v \in a(w') \text{ such that } v \Vdash \varphi \end{array}$$

We write INM for the class of intuitionistic neighbourhood models, and $\mathsf{INM} \Vdash \varphi$ if φ is valid in all intuitionistic neighbourhood models. A routine canonical model construction now proves:

Proposition 3. *For all $\varphi \in \mathscr{L}$, we have $\mathsf{IM_{Ax}} \vdash \varphi$ iff $\mathsf{INM} \Vdash \varphi$.*

Step 3. Translate between sematnics. Any IFOM-structure gives rise to an intuitionistic neighbourhood model whose worlds are given by pairs $\langle w, d \rangle$ where w is a world and $d \in D_w$. Conversely, we can transform intuitionistic neighbourhood models into IFOM-structures. This requires an additional unravelling-like construction which allows us to construct the required structure of a poset with a domain for each element. If we do so carefully, we can prove:

Proposition 4. *For all $\varphi \in \mathscr{L}$, we have $\mathsf{INF} \Vdash \varphi$ if and only if $\mathsf{FOS} \Vdash \varphi$.*

Conclusion and further work

We have given an intuitionistic monotone modal logic obtained from classical monotone modal logic via a first-order route. This opens up many avenues for further research. For example, it would be interesting to study the connection with existing intuitionistic (monotone) modal logics. Also, the first-order perspective given in this abstract may also be used to find an intuitionistic non-monotone modal logic.

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A Logical Framework for Graded Deontic Reasoning — Introducing a New Research Project

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This contribution motivates and explains project LFforGDR 10.55776/PAT2141924, recently granted by FWF. The aim of the corresponding talk is not only to inform about this endeavor and to present preliminary results, but also to elicit reactions and discuss possible co-operations within the community.

Introduction

Logical reasoning about norms, obligations and permissions is important in many areas, from philosophy, legal considerations and social decisions to software development and AI. The development of the field is driven by various problems and puzzles of informal normative reasoning. For example, it is widely recognized that the adequate formalization of reasoning with defeasible deontic conditionals requires systems of modal logics that go beyond the mere addition of a standard (monadic) modal operator O (for 'it is obligatory') to classical logic. In particular, many intended applications require a dyadic modal operator where the semantics of $\mathsf{O}(\varphi/\psi)$ ('φ should be the case if ψ holds') refers to a preference ordering for possible worlds (see [14]). Although this introduces *implicit (comparative) degrees of 'goodness'*, there is only very little research on logics with *explicitly graded deontic* propositions, so far.

Deontic propositions that allow for degrees of truth are anything but exceptional, but are probably part of the core of informal normative argumentation. The statements *'You should not kill innocent children'*, *'You should not lie'* and *'You should be polite'* are hardly appropriately categorized as *equally* true. Linguistic findings, confirm that typical deontic statements such as *'Peter should take care of Anna'*, *'Children are allowed to make noise'* or *'The place should be kept dry'* are gradable, as attested by the applicability of qualifiers such as *very much. probably* or *barely.* Moreover, the sentences that occur within a deontic modality are already typically gradable, as these examples show. Some deontic statements involve the comparison of gradations of applicability, as in *'The richer one is, the more one should donate to charity'*. For a detailed linguistic account of the gradability of *ought* and *should*, we refer to Section 8.13 of [11], where it is forcefully argued that

> In order to model this [documented linguistic] behaviour, we need to treat the basic form of *ought* as a scalar predicate associating propositions φ with "the degree to which φ ought to hold." [11], p. 249.

Another example that we will take up in Section are questionnaires and opinion polls where people are asked to indicate on a certain scale (say 0 to 10) to what extent they agree with certain deontic statements, such as *'Taxes should be more progressive'* or *'Class sizes should be drastically reduced'*. Again, note that not only the deontic propositions themselves, but also the underlying non-modal propositions can reasonably be understood to admit degrees of truth.

Recognizing that it is appropriate and useful to consider graded deontic logics for logical models of normative reasoning does not entail that it is clear how such logics should look like. Indeed, there are many different parameters to consider when defining and examining such logics. It is not always clear which deontic (monadic or dyadic) modal operators are most appropriate. If we consider relevant comparative operators, further syntactic and semantic choices emerge. For example, one might want to model sentences of the form *'It is better that F than G'* or of the form *'Given that H holds, it is just as good that F as that G'*. When generalizing to a many-valued environment, one is confronted with a variety of options for the choice of a particular base logic. First of all, it is not clear with respect to which type of scale the relative or partial truth of deontic (and other) propositions should be comparable. The space of possible algebras for modeling the degrees of truth is large and diverse, as is known from the literature on mathematical fuzzy logics (see [5]). But even if one focuses on a particular fuzzy logic with the truth value set $[0, 1]$, e.g. Łukasiewicz logic or Gödel logic, to represent non-modal sentences, there are still many ways to extend many-valued semantics to deontic modal operators that scope over them.

We argue that the above challenge to research cannot be adequately met by examining particular many-valued deontic logics individually. Rather, one should attempt to provide a *logical framework* that can be instantiated in different ways to obtain concrete logical models of reasoning with graded norms, obligations, and permissions. Specifically, the project aims to develop a comprehensive semantic and syntactic proof-theoretic *toolbox* that allows a wide range of graded deontic logics and corresponding proof systems to be systematically defined, studied and compared.

A logical framework for graded deontic reasoning

Our aim is not just to develop new graded deontic logics, but rather to systematically explore the space of possibilities for defining such logics and to evaluate the resulting logics in terms of first principles of reasoning about relative goodness and degrees of desirability and of commitments. We briefly describe essential components for such a logical framework:

Goodness relations: Classical deontic logics with a dyadic obligation operator $\mathsf{O}(\varphi/\psi)$ refer to a relation between possible worlds that orders these worlds with respect to their degree of goodness or desirability. Depending on the properties of this order and the way in which one refers to this order, different deontic logics emerge. For example, the following have been discussed in the literature for a relation $\succeq$ ('at least as good') on propositions (see, e.g., Lassiter [11]): $p \succeq q$ implies $\neg q \succeq \neg p$, $p \succeq q$ implies $p \wedge q \approx q$, or $p \succeq q$ implies $p \succeq p \vee q \succeq q$.

Relating degrees of goodness and degrees of truth: Goodness orders and corresponding degrees do not automatically translate into many-valued deontic propositions. Standard linguist approaches (see, e.g., [11]) suggest to use a context-dependent threshold value referring to the degree of goodness $\mu(\varphi)$ of a proposition φ to decide whether an assertion of φ is acceptable (as true) or not. However, it also viable to identify $\mu(\varphi)$ with a corresponding degree of truth to the proposition $\mathsf{O}(\varphi)$ ('It should be the case that φ'). For the meaningfulness and fecundity of the latter strategy we refer to an analogous approach, due to Esteva, Godo and Hàjek [16], where 'probably' is interpreted as a graded modality by identifying the degree of truth of $\Pi(\varphi)$ ('Probably φ') with the probability of an event corresponding to φ.

Choosing many-valued base logics via semantic games: There is a wide range of many-valued logics that may be used as underlying reasoning mechanism. In order to support the choice of suitable base logics we propose to employ semantic games (evaluation games) that characterize specific many-valued logics with respect to first principles about reasoning with graded propositions. The most prominent example of a semantic game of this type is Giles's game [14] characterizing Łukasiewicz logic by combining rules for the step-wise reduction of logically complex assertions to atomic assertions with an evaluation of the latter in terms of 'risk values'. Giles's game has been extended to other logics as well as to generalized (semi-fuzzy) quantifiers (see [10, 5] for an overview). We intend to extend Giles's game to include references to possible worlds as well as to goodness orders between worlds and propositions.

Incorporating uncertainty: Deontic (goodness) values are distinct from epistemic (uncertainty) values. However, as pointed out, e.g., by Lassiter [11], judgments about what should/ought to be the case are often not independent from expectations about the comparative likelihood of possible events. An important starting point for exploring the combination deontic and probabilistic reasoning is [3]. As the authors of [3] already indicate, one should consider generalizations of this approach to *dyadic* deontic logic, to other underlying base logics, and to various alternative measures of uncertainty.

Proof systems: We strive for soundness and completeness results for new logics emerging from the indicated framework. In particular, we claim that semantic games, as mentioned above, can be lifted to provability games using disjunctive states, which in turn correspond to analytic proof systems along the line of [6].

Two application areas

We decided to focus on two specific areas of applications within the project. Both are motivated by previous research of our group on that topics.

Judgment aggregation with graded deontic logics

Among the challenges for the assessment of fragmented and vague information is the systematic aggregation of opinions of many individuals. Classical Judgment Aggregation (JA) [10, 12, 13] poses the problem of finding a joint consistent collective judgment for a given agenda, modeled as a set of logically connected propositional formulas, based on individual bivalent (yes-no) judgments on the items in the agenda. Our attempts to generalize corresponding results to degree based deontic reasoning is based on two claims.

Claim A: In soliciting many opinions, one is often interested in what the individuals think *should* be done or *should* be the case.

Claim B: In soliciting opinions on—not only, but in particular—deontic propositions, it is useful and natural to allow for *degrees of assent/dissent*.

We will report on preliminary possibility and impossibility results for formal models of many-valued deontic judgment aggregation, extending recent results from [7].

Grading deontic situations

Puzzles for deontic logics typically deal with scenarios in which the described situation conflicts with a given norm set. For instance, in Forrester's paradox, the involved norm set is (1) "It is forbidden to murder (m)" and (2) "If one murders, one has to murder gently (g)". The puzzle arises in the situation, where m holds. To solve a deontic puzzle is to give a model that is consistent with the current situation and optimal with respect to the set of norms. In the case of Forrester's paradox, the solution is the situation $\{m, g\}$, which is consistent with in conflict with (1) but is consistent with m, satisfies (2), and is thus preferred over $\{m, \neg g\}$. In our context of graded deontic logic, the paradigmatic question arising from this discussion is: *How well* can a certain situation be resolved consistently with respect to a given set of norms?

We suggest to apply choice logics like QCL (see, e.g, [1]) to graded deontic scenarios to tackle this challenge. In this manner, new types of (graded) obligation operators arise. In particular, we will explore an alternative, game based semantics for the basic choice connective of QCL and for induced deontic operators, thus also providing an avenue for evaluating and expanding the applicability of our general framework.

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Varieties and quasivarieties of MV-monoids

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We investigate MV-monoids and their subvarieties. An *MV-monoid* is an algebra $\langle A, \vee, \wedge, \oplus, \odot, 0, 1\rangle$ where:

- $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice;
- $\langle A, \oplus, 0\rangle$ and $\langle A, \odot, 1\rangle$ are commutative monoids;
- $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;
- for every $x, y, z \in A$,

$$\begin{aligned}
(x \oplus y) \odot ((x \odot y) \oplus z) &= (x \odot (y \oplus z)) \oplus (y \odot z);\\
(x \odot y) \oplus ((x \oplus y) \odot z) &= (x \oplus (y \odot z)) \odot (y \oplus z);\\
(x \odot y) \oplus z &= ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z;\\
(x \oplus y) \odot z &= ((x \odot y) \oplus ((x \oplus y) \odot z)) \wedge z.
\end{aligned}$$

Every MV-algebra in the signature $\{\oplus, \neg, 0\}$ is term equivalent to an algebra that has an MV-monoid as a reduct, by defining, as standard, $1 := \neg 0$, $x \odot y := \neg(\neg x \oplus \neg y)$, $x \vee y := (x \odot \neg y) \oplus y$ and $x \wedge y := \neg(\neg x \vee \neg y)$. We study subdirectly irreducible MV-monoids and show that every subdirectly irreducible MV-monoids $\mathbf{A}$ is totally ordered and satisfies property: for all $x, y \in A$, $x \oplus y = 1$ or $x \odot y = 0$.

Furthermore, we investigate the bottom part of the lattice of subvarieties of MV-monoids, characterizing all the almost minimal varieties of MV-monoids as the varieties generated by:

- a reduct of a finite MV-chain of prime order ($\mathbf{Ł}_p^+$);
- the unique MV-monoid $\mathbf{C}_2^{\Delta}$ on the 3-element chain $0 < \varepsilon < 1$ satisfying $\varepsilon \oplus \varepsilon = \varepsilon$ and $\varepsilon \odot \varepsilon = 0$;

- the dual of $\mathbf{C}_2^{\Delta}$.

One of the main tool we used to develop the theory of MV-monoids is the categorical equivalence Γ between unit commutative ℓ-monoids and MV-monoids [1].

A *unital commutative ℓ-monoid* is an algebra $\langle M, \vee, \wedge, +, 1, 0, -1\rangle$ with the following properties:

- $\langle M, \vee, \wedge, +, 0\rangle$ is a commutative ℓ-monoid;
- $-1 + 1 = 0$;
- $-1 \leq 0 \leq 1$;
- for all $x \in M$ there is $n \in \mathbb{N}$ such that

$$\underbrace{(-1) + \cdots + (-1)}_{n \text{ times}} \leq x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Thus, the relationship between unital commutative ℓ-monoids and MV-monoids is similar to the one between abelian ℓ-groups and MV-algebras and we exploit this fact in several statements of our work.

We also present two versions of Hölder's theorem for unital commutative ℓ-monoids.

Theorem. Let $\mathbf{M}$ be a nontrivial totally ordered unital commutative ℓ-monoid. There is a unique homomorphism from $\mathbf{M}$ to $\mathbb{R}$.

Definition. A unital commutative ℓ-monoid $\mathbf{M}$ is *Archimedean* provided that, for all $x, y \in M$, if for all $n \in \mathbb{N}$ we have $nx \leqslant ny + 1$, then $x \leqslant y$.

Theorem. (Hölder's theorem for unital commutative ℓ-monoids) Let $\mathbf{M}$ be an Archimedean nontrivial totally ordered unital commutative ℓ-monoid. The unique homomorphism from $\mathbf{M}$ to $\mathbb{R}$ is injective, and so $\mathbf{M}$ is isomorphic to a subalgebra of $\mathbb{R}$.

Particular examples of MV-monoids are positive MV-algebras, i.e. the $\{\vee, \wedge, \oplus, \odot, 0, 1\}$-subreducts of MV-algebras or, equivalently, the proper subquasivariety of the variety of MV-monoids (MVM), axiomatized relatively to MVM by

$$(x \oplus z \approx y \oplus z \text{ and } x \odot z \approx y \odot z) \implies x \approx y.$$

Positive MV-algebras form a peculiar quasivariety in the sense that, albeit having a logical motivation (being the quasivariety of subreducts of MV-algebras), it is not the equivalent quasivariety semantics of any logic in the sense of [2]. In this cancellative setting, we characterized the varieties of positive MV-algebras.

Theorem. The varieties of positive MV-algebras are precisely the varieties generated by a finite set of finite positive MV-algebras. Equivalently, they are precisely the varieties generated by a finite subset of $\{\mathbf{Ł}_n^+ \mid n \in \mathbb{N}\setminus\{0\}\}$, where $\mathbf{Ł}_n^+$ is the $\{\vee, \wedge, \oplus, \odot, 0, 1\}$-reduct of the $n+1$-element MV-chain $\mathbf{Ł}_n$.

We also proved that such reducts coincide with the subdirectly irreducible finite positive MV-algebras. Using these results we show that positive MV-algebras form an unbounded sublattice of the lattice of all subvarieties of MVM.

Indeed, we prove that: a variety of positive MV-algebras is of the form $\mathscr{V}(\mathscr{K}_I)$, where I is a finite subset of $\mathbb{N}$ containing all the divisors of its elements (*divisor-closed subsets*)

Theorem. The set $\Lambda(\mathsf{MV}^+)$ of varieties of positive MV-algebras is in bijection with the set $\mathscr{I}$ of divisor-closed finite sets, as witnessed by the inverse functions:

$$f\colon \mathscr{I} \longrightarrow \Lambda(\mathsf{MV}^+) \qquad\qquad g\colon \Lambda(\mathsf{MV}^+) \longrightarrow \mathscr{I}$$
$$I \longmapsto \mathscr{V}(\mathscr{K}_I) \qquad\qquad \mathscr{V} \longmapsto \{n \in \mathbb{N}\setminus\{0\} \mid \mathbf{Ł}_n^+ \in \mathscr{V}\}.$$

where we denote by $\mathscr{K}_I$ is the set of all reducts of MV-chains with cardinality in I.

Furthermore, we present axiomatizations of all varieties of positive MV-algebras, using a strategy similar to that of Di Nola and Lettieri [3]. To do so, we define the following set of equations.

Let $I \subseteq \mathbb{N}$ be a divisor-closed set, and let m be the maximum of I (with the convention that $m = 0$ if $I = \varnothing$). We define Σ_I as the set of equations given by the single equation

$$(m+1)x \approx mx \tag{0.1}$$

union the equations of the form

$$m((k-1)x)^k \approx (kx)^m \tag{0.2}$$

for all $1 \leqslant k \leqslant m$ such that $k \notin I$.

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we define the unary term $\tau_{n,k}(x)$ inductively on n as follows:

$$\tau_{0,k}(x) := \begin{cases} 1 & \text{if } k \leqslant -1, \\ 0 & \text{if } k \geqslant 0. \end{cases}$$

$$\tau_{n+1,k}(x) = \tau_{n,k-1}(x) \odot (x \oplus \tau_{n,k}(x)),$$

For every $n \in \mathbb{N}$, let Φ_n be the following set of equations, for k ranging in $\{0, \ldots, n-1\}$:

$$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x) \text{ and } \tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x). \tag{0.3}$$

Theorem. Let I be a divisor-closed finite set; then $\mathscr{V}(\mathscr{K}_I)$ is axiomatized by $\Phi_{\mathrm{lcm}(I)} \cup \Sigma_I$

relatively to the variety of MV-monoids, where $\mathscr{K}_I$ is the set of all reducts of MV-chains with cardinality in I.

To conclude, in the following table we summarize our axiomatizations of the almost minimal varieties of MV-monoids and the varieties of positive MV-algebras.

Variety	**Axiomatization (within MV-monoids)**
$\mathscr{V}(\mathbf{C}_2^{\Delta})$	$x \oplus x \approx x$
$\mathscr{V}(\mathbf{C}_2^{\nabla})$	$x \odot x \approx x$
$\mathscr{V}(\mathbf{L}_1^{+})$	$x \oplus x \approx x$ and $x \odot x \approx x$
$\mathscr{V}(\mathbf{Ł}_n^{+})$	$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x)$ (for $0 \leqslant k \leqslant n-1$) $\tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x)$ (for $0 \leqslant k \leqslant n-1$)
$\mathscr{V}(\{\mathbf{Ł}_n^{+} \mid n \in I\})$ (I div.-closed fin. set)	(setting $l := \mathrm{lcm}(I)$ and $m := \max I$) $\tau_{l,k}(x) \oplus \tau_{l,k}(x) \approx \tau_{l,k}(x)$ (for $0 \leqslant k \leqslant l-1$) $\tau_{l,k}(x) \odot \tau_{l,k}(x) \approx \tau_{l,k}(x)$ (for $0 \leqslant k \leqslant l-1$) $(m+1)x \approx mx$ $m((k-1)x)^k \approx (kx)^m$ (for $1 \leqslant k \leqslant m$ s.t. $k \notin I$)

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The structure of the ℓ-pregroup $\mathbf{F}_n(\mathbf{Z})$

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Abstract

Lattice-ordered pregroups (ℓ-pregroups) are exactly the involutive residuated lattices where addition and multiplication coincide. Among them, for every n, the n-periodic ℓ-pregroup $\mathbf{F}_n(\mathbb{Z})$ of n-periodic order-preserving functions on $\mathbb{Z}$ plays an important role in understanding distributive ℓ-pregroups and also n-periodic ones. We study the structure of this algebra in great detail and provide order-theoretic and monoidal-theoretic descriptions. This then paves the way for axiomatizing the variety generated by $\mathbf{F}_n(\mathbb{Z})$, covered in a different submission.

Introduction

A *lattice-ordered pregroup* (*ℓ-pregroup*) is an algebra $(A, \wedge, \vee, \cdot, ^\ell, ^r, 1)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leqslant$, and for all $x \in A$,

$$x^\ell x \leqslant 1 \leqslant xx^\ell \text{ and } xx^r \leqslant 1 \leqslant x^r x.$$

We often refer to x^ℓ and x^r as the *left* and *right inverse* of x, respectively. The well-studied lattice-ordered groups (ℓ-groups) are exactly the ℓ-pregroups where the two inverses coincide: $x^\ell = x^r$. Also, ℓ-pregroups constitute lattice-ordered versions of *pregroups*, which are ordered structures introduced by Lambek [10] in the study of applied linguistics, where they are used to describe sentence patterns in many natural languages; they have also been studied extensively by Buzkowski [3] and others in the context of mathematical linguistics in connection to context-free grammars. Pregroups where the order is discrete (and also pregroups that satisfy $x^\ell = x^r$) are exactly groups.

The main reason for our interest in ℓ-pregroups is that they are precisely the *involutive residuated lattices* that satisfy $x + y = xy$; in that respect their study is connected to the algebraic semantics of *substructural logics* [8].

It is easy to show that the underlying lattices of ℓ-groups are distributive, but it remains an open problem whether every ℓ-pregroup is distributive. Partial answers to this question include [7], where it is shown that ℓ-pregroups are semidistributive, and [7], where it is shown that all *periodic* (see below) ℓ-pregroups are distributive. We denote by DLP the variety of *distributive ℓ-pregroups*.

In analogy to Cayley's theorem for groups, Holland's *embedding theorem* [9] shows that every ℓ-group can be embedded into a symmetric ℓ-group $\mathbf{Aut}(\mathbf{\Omega})$—the group of order-preserving permutations on a totally ordered set $\mathbf{\Omega}$. Also, Holland's *generation theorem* [10] states that $\mathbf{Aut}(\mathbb{Q})$ generates the variety of ℓ-groups and this is further used to show that the equational theory of ℓ-groups is decidable. In [4] it is shown that every distributive ℓ-pregroup embeds into a functional ℓ-pregroup $\mathbf{F}(\mathbf{\Omega})$ (a generalization of a symmetric ℓ-group), where $\mathbf{\Omega}$ is a chain; actually $\mathbf{\Omega}$ can be taken to be an ordinal sum of copies of the integers, as shown in [5]. Under the general definition where $\mathbf{\Omega}$ is an arbitrary chain, the algebra $\mathbf{F}(\mathbf{\Omega})$ consists of all functions on $\mathbf{\Omega}$ that have residuals and dual residuals of all orders, but in the special case where $\mathbf{\Omega}$ is the chain of the integers, $\mathbf{F}(\mathbb{Z})$ ends up consisting of all order-preserving functions on $\mathbb{Z}$ that are finite-to-one (the preimage of every singleton is a finite set/interval). This representation theorem for distributive ℓ-pregroups is used in [5] to prove an analogue of Holland's generation theorem: the ℓ-pregroup $\mathbf{F}(\mathbb{Z})$ generates the variety DLP (and that its equational theory is decidable).

For every positive integer n, the functions f in $\mathbf{F}(\mathbb{Z})$ that are periodic and have period n end up being exactly the ones that satisfy $f^{\ell^n} = f^{r^n}$ and they form a subalgebra of $\mathbf{F}(\mathbb{Z})$, which we denote by $\mathbf{F}_n(\mathbb{Z})$; here $f^{\ell^3} = f^{\ell\ell\ell}$, for example. In [6] it is proved that DLP is equal to the join of the varieties $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$. This demonstrates the importance of the varieties $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, and hence also the algebras $\mathbf{F}_n(\mathbb{Z})$, in understanding distributive ℓ-pregroups. For example, if an equation fails in DLP, it fails in some $\mathbf{F}_n(\mathbb{Z})$ (and [6] further provides a concrete suitable n).

More generally, in an arbitrary ℓ-pregroup an element x is called *n-periodic* if $x^{\ell^n} = x^{r^n}$; an ℓ-pregroup is called *n-periodic* if all of its elements are, and the corresponding variety is denoted by LP_n. As mentioned before, in [7] it is shown that $\mathsf{LP}_n \subseteq \mathsf{DLP}$, for all n, and in [6] it is further proved that the join of all of the LP_n's is exactly DLP. Thus $\mathsf{DLP} = \bigvee \mathsf{LP_n} = \bigvee \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$. These two appoximations of DLP are quite different since, as shown in [6], the variety $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ is properly contained in $\mathsf{LP_n}$ for every single n. Even though $\mathsf{LP_n} \neq \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, for every n, $\mathbf{F}_n(\mathbb{Z})$ actually plays an important role in understanding LP_n, as well: it is shown in [6] that every n-periodic ℓ-pregroup can be embedded in a *wreath product* of an ℓ-group and $\mathbf{F}_n(\mathbb{Z})$.

The structure of the algebra

In [6], enough aspects of $\mathbf{F}_n(\mathbb{Z})$ are studied in order to obtain the above results and also the decidability of the equational theory of $\mathbf{F}_n(\mathbb{Z})$, for all n. However, the lattice-theoretic and monoidal-theoretic structure of $\mathbf{F}_n(\mathbb{Z})$ has been described only for $n = 2$, in [7]. In this contribution we provide a detailed description of $\mathbf{F}_n(\mathbb{Z})$, for all n.

Toward describing the monoidal structure of $\mathbf{F}_n(\mathbb{Z})$ we first identify two of its submonoids: $\boldsymbol{b}^{\mathbb{Z}}$ and $\mathbf{End}(\mathbf{n})$. We denote by $\mathbf{b}$ the function $x \mapsto x + 1$ on $\mathbb{Z}$ and by $\boldsymbol{b}^{\mathbb{Z}} := \{\mathbf{b}^k : k \in \mathbb{Z}\}$

the subgroup that it generates; $\boldsymbol{b}^{\mathbb{Z}}$ is an ℓ-group (isomorphic to $\mathbb{Z}$) and it is the largest subgroup/ℓ-subgroup of $\mathbf{F}_n(\mathbb{Z})$ (i.e., the set of all of invertible elements of $\mathbf{F}_n(\mathbb{Z})$). For each n the maps

$$\sigma_n(x) = x \wedge x^{\ell\ell} \wedge \cdots \wedge x^{\ell^{2n-2}} \text{ and } \gamma_n(x) = x \vee x^{\ell\ell} \vee \cdots \vee x^{\ell^{2n-2}},$$

are an interior and a closure operator on $\mathbb{Z}$, respectively; also they both have image equal to $\boldsymbol{b}^{\mathbb{Z}}$. In particular, $\mathbf{F}_n(\mathbb{Z})$ is the convex closure of $\boldsymbol{b}^{\mathbb{Z}}$.

For every $n \in \mathbb{Z}$, we denote by $\mathbf{n}$ the n-element chain $0 < 1 < \ldots < n-1$ and by $End(\mathbf{n})$ the endomorphisms (i.e., order-preserving maps) on $\mathbf{n}$. $End(\mathbf{n})$ forms a distributive lattice by pointwise order and a monoid under functional composition, and multiplication distributes over both join and meet; the resulting *distributive lattice-ordered monoid* (DLM) is denoted by $\mathbf{End}(\mathbf{n})$.

The n-periodic extensions to $\mathbb{Z}$ of the functions in $\mathbf{End}(\mathbf{n})$ form a subDLM of $\mathbf{F}_n(\mathbb{Z})$, which we denote this subDLM of $\mathbf{F}_n(\mathbb{Z})$ by $\mathbf{End}(\mathbf{n})$ as well, by abusing notation. We observe that actually this is only one of the n-many different (overlapping) subDLM of $\mathbf{F}_n(\mathbb{Z})$ that are isomorphic to $\mathbf{End}(\mathbf{n})$. We prove that the union of these copies of $\mathbf{End}(\mathbf{n})$ in $\mathbf{F}_n(\mathbb{Z})$ is equal to the set of flat elements of $\mathbf{F}_n(\mathbb{Z})$, and is contained in the interval $[\mathbf{b}^{1-n}, \mathbf{b}^{n-1}]$; an element x of an ℓ-pregroup is called *flat* if there exist idempotents y, z such that $y \leqslant x \leqslant z$.

We prove that every element of $\mathbf{F}_n(\mathbb{Z})$ can be written as a product of the form xy and of the form $y'x'$, where $x, x' \in End(\mathbf{n})$ and $y, y' \in \mathbf{b}^{\mathbb{Z}}$; thus, $F_n(\mathbb{Z}) = End(\mathbf{n}) \cdot \mathbf{b}^{\mathbb{Z}} = \mathbf{b}^{\mathbb{Z}} \cdot End(\mathbf{n})$. Furthermore, we prove that each of these decompositions is unique. However, $\mathbf{F}_n(\mathbb{Z})$ is not isomorphic to the direct product of $\mathbf{End}(\mathbf{n})$ and $\mathbb{Z}$, nor even to a semidirect product of them.

Let $\mathbf{M}$ and $\mathbf{N}$ be monoids, where

- $*\colon M \times N \to N$ a left action of $\mathbf{M}$ on $\mathbf{N}$: $1 * n = n$ and $m_1 * (m_2 * n) = m_1 m_2 * n$,
- $\star\colon M \times N \to M$ a right action of $\mathbf{N}$ in $\mathbf{M}$: $m \star 1 = m$ and $(m \star n_1) \star n_2 = m \star n_1 n_2$,
- $m * n_1 n_2 = (m * n_1)((m \star n_1) * m_2)$ and
- $m_1 m_2 \star n = (m_1 \star (m_2 * n))(m_2 \star n)$.

Then $\mathbf{N} \times^*_\star \mathbf{M} = \langle N \times M, \circ, \langle 1, 1 \rangle \rangle$, where $\langle n_1, m_1 \rangle \circ \langle n_2, m_2 \rangle = \langle n_1(m_1 * n_2), (m_1 \star n_2) m_2 \rangle$, is a monoid called the Zappa product of $\mathbf{N}$ and $\mathbf{M}$ with respect to the two actions. Note that if $\star$ is trivial ($m \star n = m$, for all n, m) or $*$ is trivial, then the Zappa product is a (left or right) semidirect product.

Theorem 1. *The monoid reduct of $\mathbf{F}_n(\mathbb{Z})$ is isomorphic to the Zappa product $\mathbf{End}(\mathbf{n}) \times^*_\star \mathbb{Z}$, where $b^m \star a = c$ and $a * b^m = b^k$, and $c \in \mathbf{End}(\mathbf{n})$ and $k \in \mathbb{Z}$ are the unique elements such that $b^m a = c b^k$.*

Now, to describe the order structure of $\mathbf{F}_n(\mathbb{Z})$ we describe its poset of join irreducibles. We define the poset $\mathbf{C}_n^{\mathbb{Z}} := ([0, n-1] \times \mathbb{Z}, \leqslant)$ by: for $(m, k), (m', k') \in [0, n-1] \times \mathbb{Z}$,

$$(m, k) \leqslant (m', k') :\Longleftrightarrow\ m -_n m' \leqslant (k' - m') - (k - m).$$

As usual for $m, m' \in [0, n-1]$, $m -_n m'$ (difference modulo n) is equal to $m - m'$ if $m \geqslant m'$ and to $m - m' + n$ if $m < m'$. In particular, $m -_n m' \in [0, n-1]$. Since $0 \leqslant m -_n m'$, this definition implies $k - m \leqslant k' - m'$. Also, we note that the corresponding covering relation is

$$(m, k) \prec (m', k') :\Longleftrightarrow\ m + 1 = m' \text{ or } (m = m' \text{ and } k = k' +_n 1).$$

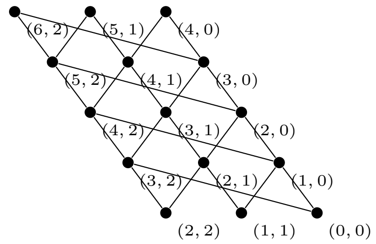

Figure 1: The infinite-layered posets $\mathbf{C}_3^{\mathbb{Z}}$ and $\mathbf{C}_4^{\mathbb{Z}}$.

Theorem 2. *An element of $\mathbf{F}_n(\mathbb{Z})$ is join irreducible iff it is meet irreducible. Also, the poset of join irreducibles of $\mathbf{F}_n(\mathbb{Z})$ is isomorphic to $\mathbf{C}_n^{\mathbb{Z}}$.*

We further define a multiplication on $\mathbf{C}_n^{\mathbb{Z}}$ by:

$$(m', k') \cdot (m, k) = (m, S_n(k - m') + k').$$

Here, for every $n, a \in \mathbb{Z}$, we define $S_n(a) := qn$, where $a = qn + r$, $0 \leqslant r < n$ and $q, r \in \mathbb{Z}$ are given by the division algorithm; i.e., $S_n(a)$ is the largest whole multiple of n below (and including) a. Note that $(\mathbf{C}_n^{\mathbb{Z}}, \cdot)$ is a semigroup isomorphic to the semidirect product of $(\mathbb{Z}, +)$ and the right-zero semigroup on $[0, n-1]$, where the action $* : [0, n-1] \times \mathbb{Z} \to \mathbb{Z}$ is defined by $m * k := S_n(k - m)$.

Theorem 3. *The join irreducibles of $\mathbf{F}_n(\mathbb{Z})$ form a partially-ordered semigroup that is isomorphic to $(\mathbf{C}_n^{\mathbb{Z}}, \leqslant, \cdot)$. Also, the join irreducibles of $\mathbf{F}_n(\mathbb{Z})$ are closed under the inverses, and the corresponding operations on $\mathbf{C}_n^{\mathbb{Z}}$ are:*

$$(m, k)^{\ell} := (k{-}n{+}1{-}S_n(k{-}n{+}1), m{-}S_n(k{-}n{+}1)) \text{ and } (m, k)^{r} := (k{-}S_n(k), m{+}n{-}1{-}S_n(k)).$$

In view of this result the elements of $\mathbf{F}_n(\mathbb{Z})$ can be viewed as downsets of $\mathbf{C}_n^{\mathbb{Z}}$; then the lattice operations are simply union and intersection, while multiplication and the inversions is simply the element-wise lifting of the multiplication and inversions on $\mathbf{C}_n^{\mathbb{Z}}$.

In view of Theorem 1 we also provide an analysis of $\mathbf{End(n)}$ as a DLM, and of the positive cones of $\mathbf{End(n)}$ and $\mathbf{F}_n(\mathbb{Z})$, via their poset of join irreducibles (as a finitary or one-sided versions of $\mathbf{C}_n^{\mathbb{Z}}$), as well as their multiplicative structure in terms of irreducible generators.

The above analysis can be used to prove the following generation result. The *periodicity* of an element is defined to be the smallest positive k such that the element is k-periodic.

Theorem 4. $\mathbf{F}_n(\mathbb{Z})$ *is generated by any one of its elements of periodicity n.*

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Decidability of Bernays–Schönfinkel Class of Gödel Logics

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It is widely acknowledged that any first-order formula in classical logic is logically identical to one in prenex form. In general, any set of quantifier prefixes defines a fragment of first-order logic, specifically the set of prenex formulas that contain one of the quantifier prefixes in question. In the early stages of research, it was recognised that while some fragments defined in this way have decidable satisfiability/validity, others do not.

In 1928, P. Bernays and M. Schönfinkel proved the decidability for the class of function-free sentences with prefixes $\exists\bar{x}\forall\bar{y}A(\bar{x},\bar{y})$ (satisfiability) and $\forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ (validity) (specifically, the set of sentences that, when written in prenex normal form, have a prefix containing quantifiers and the matrix without function symbols) [5]. We will study the decidability of the Bernays–Schönfinkel class for all Gödel logics. Our argument for validity is based on the fact that Skolemization is possible for prenex Gödel logics and our argument for satisfiability is based on the general properties of prenex formulas. We must note that in Gödel logics validity and satisfiability are not dual as in classic logic.

Definition 1. (Gödel logics). First-order Gödel logics are a family of many-valued logics where the truth values set (known also as *Gödel set*) V is closed subset of the full $[0,1]$ interval that includes both 0 and 1 given by the following evaluation function $\mathscr{I}$ on V

$$
\begin{aligned}
&(1) \quad \mathscr{I}(\bot) = 0\\
&(2) \quad \mathscr{I}(A \wedge B) = min\{\mathscr{I}(A), \mathscr{I}(B)\}\\
&(3) \quad \mathscr{I}(A \vee B) = max\{\mathscr{I}(A), \mathscr{I}(B)\}\\
&(4) \quad \mathscr{I}(A \supset B) = \begin{cases} \mathscr{I}(B) & \text{if } \mathscr{I}(A) > \mathscr{I}(B), \\ 1 & \text{if } \mathscr{I}(A) \leq \mathscr{I}(B). \end{cases}\\
&(5) \quad \mathscr{I}(\forall x A(x)) = \inf\{\mathscr{I}(A(u)) : u \in U_{\mathscr{I}}\}\\
&(6) \quad \mathscr{I}(\exists x A(x)) = \sup\{\mathscr{I}(A(u) : u \in U_{\mathscr{I}}\}
\end{aligned}
$$

Definition 2. (1-entailment). For a truth value set V, a (possibly infinite) set Γ of formulas (1-)entails a formula A if the interpretation $\mathscr{I}$ on V of A is 1 in case the interpretations of all formulas in Γ are 1, i.e., $\Gamma \Vdash_V A \Longleftrightarrow (\forall \mathscr{I}, \forall B \in \Gamma : \mathscr{I}(B) = 1) \to \mathscr{I}(A) = 1$.

As a generalization of classical satisfiability, we introduce the following concepts:

Definition 3 (Validity)**.** The formula in Gödel logic is *valid* if the formula evaluates to 1 under every interpretation.

Definition 4 (satisfiability)**.** The formula in Gödel logic is *1-satisfiable* if there exists at least one interpretation that assigns 1 to the formula.

Validity in Berneys-Schönfinkel class for all Gödel logics is Decidable

Definition 5 (Structural Skolem form)**.** Let A be a closed first-order formula. Whenever A does not contain strong quantifiers, we define its *structural Skolem form* as $A^S = A$. Suppose now that A contains strong quantifiers. Let (Qy) be the first strong quantifier occurring in A. If (Qy) is not in the scope of weak quantifiers, then its structural Skolem form is

$$A^S = (A_{-(Qy)}\{y \leftarrow c\})^S,$$

where $A_{-(Qy)}$ is the formula A after omission of (Qy) and c is a constant symbol not occurring in A. If (Qy) is in the scope of the weak quantifiers $(Q_1x_1)\ldots(Q_nx_n)$, then its structural Skolemization is

$$A^S = (A_{-(Qy)}\{y \leftarrow f(x_1, \ldots, x_n)\})^S,$$

where f is a function symbol (Skolem function) and does not occur in A.

In Gödel logics, valid prenex formulas can be sharpened to validity equivalent purely existential formulas by Skolemization.

Lemma 1. *(Skolemization) For all prenex formulas $Q\bar{x}A(\bar{x})$ and all Gödel logics G*

$$\Gamma \Vdash_G Q\bar{x}A(\bar{x}) \Longleftrightarrow \Gamma \Vdash_G (Q\bar{x}A(\bar{x}))^S$$

where $Q\bar{x}$ is a quantifier prefix and $A(\bar{x})$ is a quantifier-free formula.

Proof. It is sufficient to prove with A arbitrary and f a new function:

$$\Gamma \Vdash_G \exists\overline{x}\forall y A(\overline{x}, y) \Leftrightarrow \Gamma \Vdash_G \exists\overline{x}A(\overline{x}, f(\overline{x})).$$

It follows then from induction. ($\Rightarrow$) The direction from left to right is obvious.
($\Leftarrow$) For the other direction, if $\nVdash_G \exists\overline{x}\forall y A(\overline{x}, y)$ then for some interpretation $\mathscr{I}$

$$\supseteq \{d_{\overline{c}} \mid \mathscr{I}(\forall y A(\overline{c}, y)) = d_{\overline{c}}\} \leqslant d < 1.$$

Using the axiom of choice we can assign a value for every $f(\bar{c})$ such that $\mathscr{I}(A(\bar{c}, f(\bar{c}))$ is in between $d_{\bar{c}}$ and $d_{\bar{c}} + \frac{1-d}{2}$. As a consequence

$$\supseteq \{d_{\bar{c}} + \frac{1-d}{2} \mid \mathscr{I}(A(\bar{c}, f(\bar{c}))) \leqslant d_{\bar{c}} + \frac{1-d}{2}\} \leqslant d + \frac{1-d}{2} < 1$$

and thus $\Gamma \nVdash_G \exists \bar{x} A(\bar{x}, f(\bar{x}))$. □

Theorem 1. *Validity in Berneys-Schönfinkel (BS) class is decidable for all Gödel logics.*

Proof. from above lemma follows

$$\Gamma \Vdash_G \forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y}) \Longleftrightarrow \Gamma \Vdash_G \exists \bar{y} A(\bar{c}, \bar{y})$$

for new constants $\bar{c}$. Suppose there is a countermodel M such that $M \nVdash_G \exists \bar{y} A(\bar{c}, \bar{y})$. Then there is also a countermodel M' such that $M' \nVdash_G \exists \bar{y} A(\bar{c}, \bar{y})$ where the domain of M' contains only interpretations of $\bar{c}$. □

Corollary 1. *1) Let $\exists \bar{y} A(\bar{y})$ contain only constants $\bar{c}$, then Herbrand's theorem holds for $\exists \bar{y} A(\bar{y})$ for all Gödel logics G.*

2) Let $\forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$ prenex formulas contain only constants $\bar{d}$, then $\Gamma \Vdash_G \forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y}) \Longleftrightarrow \Gamma \Vdash_{G'} \forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$ for all infinitely-valued Gödel logics G, G'.

Proof. 1) According to the proof of the above theorem, $M \nVdash_G \exists \bar{y} A(\bar{c}, \bar{y})$ implies $M' \nVdash_G \exists \bar{y} A(\bar{c}, \bar{y})$ with restricted domain to constants only.
2) follows from 1), as Herbrand disjunction is contained in $\bigvee_n A(\bar{c}_n, \bar{d}_n)$ where $\bar{c}_n, \bar{d}_n$ are possible variations of $\bar{c}, \bar{d}$ and validity for propositional formulas coincides with infinitely-valued Gödel logics. □

Remark 1. Note that 1) is not trivial as prenex formulas and consequently $\exists$-formulas (see. Lemma 1) for countable Gödel logics are not r.e.[1].

1-satisfibility in Berneys-Schönfinkel class for all Gödel logics is Decidable

Lemma 2 (Gluing lemma)**.** *Let $\mathscr{I}$ be an interpretation into $V \subseteq [0,1]$. Let us fix a value $\omega \in [0,1]$ and define*

$$\mathscr{I}_\omega(\mathscr{P}) = \begin{cases} \mathscr{I}(\mathscr{P}) & \text{if } \mathscr{I}(\mathscr{P}) \leqslant \omega, \\ 1 & \text{otherwise} \end{cases}$$

for atomic formula $\mathscr{P}$ in $\mathscr{L}^{\mathscr{I}}$. Then $\mathscr{I}_\omega$ is an interpretation into V such that

$$\mathscr{I}_\omega(\mathscr{B}) = \begin{cases} \mathscr{I}(\mathscr{B}) & \text{if } \mathscr{I}(\mathscr{B}) \leqslant \omega, \\ 1 & \text{otherwise} \end{cases}$$

As an immediate consequence, we have:

Corollary 2. *Prenex formulas in Gödel logics admit* 1*-satisfiability iff they are classical saitisfiable.*

Theorem 2. *1-satisfiability in Berneys-Schönfinkel class is decidable for all Gödel logics.*

Proof. The proof is obvious as 1-satisfiability coincides with classical satisfiability and, therefore, is decidable. □

Remark 2. All Gödel logics coincide for the Bernays-Schönfinkel class w.r.t. 1-satisfiability, but only the infinitely valued Gödel logics coincide for the Bernays-Schönfinkel class w.r.t. to validity. The Berneys-Schönfinkel fragment of any infinitely-valued Gödel logic is the intersection of the Bernays-Schönfinkel fragments of the finitely-valued Gödel logic, both for satisfiability and validity.

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Axiomatising non falsity and threshold preserving variants of MTL logics

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Fuzzy logics are logics of *graded truth* that have been proposed as a suitable tool for reasoning with imprecise information, in particular for reasoning with propositions containing vague predicates. Their main feature is that they allow to interpret formulas in a linearly ordered scale of truth-values, and this is specially suited for representing the gradual aspects of vagueness. In particular, systems of fuzzy logic have been in-depth developed within the frame of mathematical fuzzy logic [3] (MFL). Most well known and studied systems of mathematical fuzzy logic are the so-called *t-norm based fuzzy logics*, corresponding to formal many-valued calculi with truth-values in the real unit interval $[0, 1]$ and with a conjunction and an implication interpreted respectively by a (left-) continuous t-norm and its residuum, and thus, including e.g. the well-known Łukasiewicz and Gödel infinitely-valued logics, corresponding to the calculi defined by Łukasiewicz and min t-norms respectively. The most basic t-norm based fuzzy logic is the logic MTL (monoidal t-norm based logic) introduced in [6].

In logical systems in MFL, the usual notion of deduction is defined by requiring the preservation of the truth-value 1 (full *truth-preservation*), which is understood as representing the absolute truth. For instance, let L be any extension of MTL, which we assume to be complete w.r.t. the family $\mathscr{C}_L = \{[\mathbf{0}, \mathbf{1}]_* \mid [\mathbf{0}, \mathbf{1}]_* \text{ is a L-algebra}\}$ of standard L-algebras. Then the typical notion of logical consequence is the following for every set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \models_L \varphi \quad \text{if,} \quad \text{for any } [0,1]_* \in \mathscr{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \text{ if } e(\psi) = 1 \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) = 1 \text{ as well.}$$

In [2], Bou, Esteva et al. introduced the degree preserving MTL-logics where they change the (full) truth paradigm to the degree preserving paradigm, in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is greater or equal than those of the premises. For any extension L of MTL complete w.r.t. the family $\mathscr{C}_L$ of standard L-algebras the degree preserving variant of L, denoted by $L^{\leqslant}$ is defined as

$\Gamma \models_L^{\leqslant} \varphi$ if, for any $[0,1]_* \in \mathscr{C}_L$, any $[0,1]_*$-evaluation e and for any $a \in [0,1]$, if $e(\psi) \geqslant a$ for any $\psi \in \Gamma$, then $e(\varphi) \geqslant a$.

As a matter of fact, the degree preserving logic $L^{\leqslant}$ is strongly related to the 1-preserving logic L. Indeed, on the one hand, it holds that $\models_L^{\leqslant} \varphi$ iff $\models_L \varphi$, so both logics share the set of valid formulas. Moreover, if for any finite set of formulas Γ we let $\Gamma^{\wedge} = \wedge\{\psi \mid \psi \in \Gamma\}$, we can observe that

$$\Gamma \models_L^{\leqslant} \varphi \text{ iff } \models_L \Gamma^{\wedge} \to \varphi,$$

and hence, iff $\models_L^{\leqslant} \Gamma^{\wedge} \to \varphi$. This property can be seen as a sort of deduction theorem for $\models_L^{\leqslant}$.

It has been shown in [2] that in the case the logic L has a complete axiomatisation with Modus Ponens as the only inference rule, then the logic Ł$^{\leqslant}$ admits a complete axiomatisation as well, having as axioms the axioms of L and as inference rules the rule of adjunction:

$$\text{(Adj)} \quad \frac{\varphi, \quad \psi}{\varphi \wedge \psi},$$

and the following restricted form of the Modus Ponens rule

$$\text{(r-MP)} \quad \frac{\varphi, \quad \varphi \to \psi}{\psi}, \quad \text{if } \vdash_L \varphi \to \psi.$$

If the logic L has additional inference rules

$$(\mathrm{R}_i) \quad \frac{\Gamma_i}{\varphi}$$

for $i \in I$, then [4, Proposition 1] shows that $L^{\leqslant}$ is axiomatised with the above axioms and rules together with the following restricted forms of the rules (R_i):

$$(\text{r-R}_i) \quad \frac{\Gamma_i}{\varphi}, \quad \text{if } \vdash_L \Gamma_i.$$

Still, another way of defining different variants of a fuzzy logic is put forward in [1], although for the particular case of Łukasiewicz fuzzy logic. In this approach, the notion of consequence at work is the *non-falsity preservation*, according to which a conclusion follows from a set of premises whenever if the premises are non-false, so must be the conclusion. In other words, assuming a $[0,1]$-valued semantics, this is the case when, for any evaluation, if truth degrees of the premises are above 0, then the truth-degree of the conclusion is so as well. For any extension L of MTL complete w.r.t. the family $\mathscr{C}_L$ of standard L-algebras we define the following non falsity preserving variant:

$\Gamma \models_L^{(0} \varphi$ if, for any $[0,1]_* \in \mathscr{C}_L$ and any $[0,1]_*$-evaluation e , if $e(\psi) > 0$ for any $\psi \in \Gamma$, then $e(\varphi) > 0$.

The purpose of this talk is to obtain a similar type of axiomatisations for some non-falsity preserving logics. First observe that for any truth preserving logic L with standard semantics and for any formula φ it is obvious that $\models_L \varphi$ implies $\models_L^{(0} \varphi$, so the set of valid formulas of L is contained in the set of valid formulas of the non falsity preserving variant $L^{(0}$. Moreover the finitary versions of both logics are strongly related.

Lemma 1. *For every pair of formulas φ, ψ the following relation holds:*

$$\varphi \models_L^{(0} \psi \quad \textit{iff} \quad \neg\psi \models_L \neg\varphi.$$

We now focus on logics defined by classes of standard IMTL-algebras (standard MTL-algebras with an involutive negation). We remind that this means that $*$ is a left-continuous t-norm such that the residual negation $\neg$, defined as $\neg x = x \to 0 = \supseteq \{y \in [0,1] \mid x * y = 0\}$ satisfies the involutivity condition $\neg(\neg x) = x$. Notable examples of such t-norms are Łukasiewicz t-norm (which is continuous) and Nilpotent Minimum t-norm.

Assume L is an axiomatic extension of IMTL, complete w.r.t. a class of standard algebras $\mathscr{C}_L$, and whose corresponding notion of proof is denoted $\vdash_L$. It is immediate to observe that in the case of a IMTL logic L, Lemma 1 can be strengthened in the sense that the 1-preserving logic L and the non-falsity preserving logic $L^{(0}$ become interdefinable. Namely,

$$\text{(i) } \varphi \models_L \psi \text{ iff } \neg\psi \models_L^{(0} \neg\varphi, \qquad \text{(ii) } \varphi \models_L^{(0} \psi \text{ iff } \neg\psi \models_L \neg\varphi \ .$$

In order to syntactically characterise $\models_L^{(0}$, the following system nf-L, called the *non-falsity preserving companion* of L, is defined in [5] as follows.

Definition 1. The calculus nf-L is defined by the following axioms and rules:

- Axioms of L
- Rule of Adjunction: (Adj) $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- Reverse Modus Ponens: (MP^r) $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee \ \neg(\varphi \to \psi) \ \vee \ \chi}$
- Restricted Modus Ponens: (r-MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$, if $\vdash_L \varphi \to \psi$

The above (MP^r) rule captures the following form of reverse of modus ponens: if $\neg\psi$ is non-false then either $\neg\varphi$ is non-false or $\neg(\varphi \to \psi)$ is non-false. The addition of the disjunct χ both in the premise and in the conclusion of the rule is needed for technical reasons.

The following is a syntactic counterpart of part of Lemma 1.

Proposition 1. *If $\psi \vdash_L \varphi$ then $\neg\varphi \vdash_{\mathsf{nf}\text{-}L} \neg\psi$.*

Thanks to this relation, the logic nf-L has been shown to be complete with respect to the intended semantics.

Theorem 1. *Let L be an axiomatic extension of IMTL. Then, the calculus* nf-*L is sound and complete w.r.t. the finitary logic of* $L^{(0}$.

Note that, as a direct corollary, Definition 1 provides us with complete axiomatisations of non-falsity preserving companions of prominent IMTL logics like Łukasiewicz logic or Nilpotent Minimum logic.

We are also able to prove similar result as in the previous theorem without the requirement of the negation $\neg$ to be involutive. Indeed, let $\mathrm{MTL}_{\neg\neg}$ be the (non-axiomatic) extension of MTL with the rule

$$(\mathrm{R}_{\neg\neg}) \quad \frac{\neg\neg\varphi}{\varphi}.$$

The algebraic semantics of $\mathrm{MTL}_{\neg\neg}$ consists of the quasi-variety generated by the class of MTL-chains $\mathbf{A}$ such that its negation $\neg$ is such that, for any $a \in A$, $\neg a = 0$ iff $a = 1$, or equivalently $\neg a > 0$ iff $a < 1$. If L is an axiomatic extension of MTL, let us denote by $\mathrm{L}_{\neg\neg}$ the extension of L with the rule $(\mathrm{R}_{\neg\neg})$. If L is complete w.r.t. a class of standard algebras $\mathscr{C}_L$, then $L_{\neg\neg}$ is also complete w.r.t. the class of standard algebras $\mathscr{C}_{L_{\neg\neg}}$. Moreover, in $L_{\neg\neg}$ we keep having at the semantical level the equivalence between the 1-preserving logic and the non-falsity preserving logic, in the following sense.

Lemma 2. *For any fuzzy logic L, then the following conditions hold:*

$$(i)\ \varphi \models_{L_{\neg\neg}} \psi \text{ iff } \neg\psi \models^{(0}_{L_{\neg\neg}} \neg\varphi, \qquad (ii)\ \varphi \models^{(0}_{L_{\neg\neg}} \psi \quad \text{iff} \quad \neg\psi \models_{L_{\neg\neg}} \neg\varphi.$$

Then one can define the non-falsity preserving companion of a $\mathrm{MTL}^{\neg\neg}$-logic and prove its completeness as follows. In fact, we can restrict ourselves to extensions of MTL logics with the rule $(\mathrm{R}_{\neg\neg})$. where $\neg(\varphi \wedge \neg\varphi)$ is not a tautology, that is extensions of non SMTL-logics with the rule $(\mathrm{R}_{\neg\neg})$. Indeed, note that if L is an SMTL logic, then $L_{\neg\neg}$ collapses into classical logic.

Theorem 2. *Let L be an axiomatic extension of MTL that is non-SMTL. Then the calculus* nf-$L_{\neg\neg}$*, defined by the following axioms and rules:*

- *Axioms of L*
- *The rule* $(R_{\neg\neg})$
- *The rule of adjunction (Adj)*
- *The rule of Reverse Modus Ponens* (MP^r)
- *The rule of Restricted Modus Ponens (r-MP)*

is a sound and complete axiomatisation w.r.t. to the finitary logic of $L^{(0}_{\neg\neg}$.

Finally, we turn our attention to logics preserving lower bounds of truth-values. Let L be an extension (or expansion) of MTL complete w.r.t. some class of standard L-algebras

$\mathscr{C}_L$, fix some positive value $a \in (0,1]$, we define the logic L^a as follows:

$$\Gamma \models^a_L \varphi \quad \text{if,} \quad \begin{array}{l}\text{for any } [0,1]_* \in \mathscr{C}_L, \text{ any } [0,1]_*\text{-evaluation } e,\\ \text{if } e(\psi) \geqslant a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) \geqslant a.\end{array}$$

We will end the talk by discussing some general but sufficient assumptions on L to guarantee a finitary axiomatisation of L^a.

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Amalgamation failures in MTL-algebras

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In this contribution we present some new results concerning the deductive interpolation property in substructural logics, via the study of amalgamation in their equivalent algebraic semantics which are classes of residuated lattices. In particular, we solve a long-standing open problem, showing that the following varieties do not have the amalgation property: MTL-algebras and its involutive and pseudocomplemented subvarieties, IMTL and SMTL.

Residuated structures play an important role in the field of algebraic logic; their equivalent algebraic semantics, in the sense of Blok and Pigozzi [1], encompass many of the interesting nonclassical logics: intuitionistic logic, intermediate logics, many-valued logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics.

Let us be more precise; a residuated lattice is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ of type $(2, 2, 2, 2, 2, 0)$ such that: $(A, \vee, \wedge)$ is a lattice; $(A, \cdot, 1)$ is a monoid; the residuation law holds: for all $x, y, z \in A$, $x \cdot y \leqslant z \Leftrightarrow y \leqslant x\backslash z \Leftrightarrow x \leqslant z/y$, (where $\leqslant$ is the lattice ordering). Residuated lattices form a variety. A residuated lattice is said to be: *integral* if the monoidal identity is the top element of the lattice; *commutative* if the monoidal operation is commutative; n-potent if it holds that $x^n = x^{n+1}$; *semilinear* if it is a subdirect product of chains.

Residuated lattices with an extra constant 0 are called FL-algebras (since they are the equivalent algebraic semantics of the Full Lambek calculus, see [8]), and we call them 0-*bounded* if $0 \leqslant x$; *bounded* if they are integral and 0-bounded. Semilinear bounded commutative FL-algebras are called MTL-algebras since they are the equivalent algebraic semantics of the monoidal t-norm based logic MTL [2]. Among the most relevant subvarieties of MTL-algebras we have: IMTL-algebras, given by involutive algebras ($\neg\neg x = x$), and SMTL-algebras, i.e. the pseudocomplement subclass ($x \wedge \neg x = 0$).

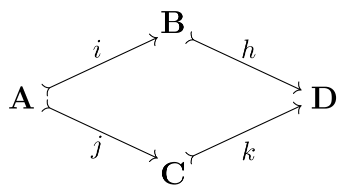
Our results use one of the most interesting *bridge theorems* that are a consequence of algebraizability: the connection between logical interpolation properties and algebraic amalgamation properties. We say that a logic $\mathscr{L}$, associated to a consequence relation $\vdash$, has the *deductive interpolation property* if for any set of formulas $\Gamma \cup \{\psi\}$, if $\Gamma \vdash \psi$ then there exists a formula δ such that $\Gamma \vdash \delta$, $\delta \vdash \psi$ and the variables appearing in δ belong to the intersection of the variables appearing both in Γ and in ψ, in symbols

$Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$.

If the logic $\mathscr{L}$ has a variety V as its equivalent algebraic semantics, and V satisfies the *congruence extension property* (CEP), $\mathscr{L}$ has the deductive interpolation property if and only if V has the *amalgamation property* (without the CEP, the amalgamation property corresponds to the stronger Robinson property, see [8]).

Let us then recall the other necessary notions.

Definition 1. Given a class K of algebras in the same signature, a *V-formation* is a tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ and i, j are embeddings of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$ respectively; an *amalgam* in K for the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ is a triple $(\mathbf{D}, h, k)$ where $\mathbf{D} \in \mathsf{K}$ and h and k are embeddings of respectively $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$ such that $h \circ i = k \circ j$.

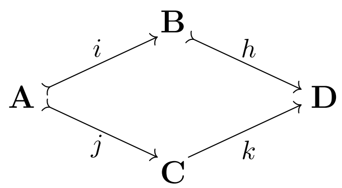


A class K of algebras has the *amalgamation property* if for any V-formation in K there is an amalgam in K.

We focus on the study of the amalgamation property in semilinear varieties of residuated lattices, solving some long-standing open problems; most importantly, we establish that semilinear commutative (integral) residuated lattices and their 0-bounded versions do not have the amalgamation property (i.e., MTL-algebras and their 0-free subreducts).

In order to obtain a failure of the amalgamation property, we use the recent results in [4]; the authors show that in a variety V with the CEP and whose class of finitely subdirectly irreducible members $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras, the amalgamation property of the variety is equivalent to the so-called *one-sided amalgamation property* of $\mathsf{V}_{\mathrm{FSI}}$.

Definition 2. Given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$, a *one-sided amalgam* for it is a triple $(\mathbf{D}, h, k)$ with $\mathbf{D} \in \mathsf{K}$ and as for amalgamation $h \circ i = k \circ j$, but while h is an embedding, k is a homomorphism.



A class K of algebras has the one-sided amalgamation property if for any V-formation there is a one-sided amalgam in K.

The mentioned result of [4] is particularly useful in varieties generated by commutative residuated chains; indeed, all commutative residuated lattices have the CEP and a semilinear residuated lattice is finitely subdirectly irreducible if and only if it is totally ordered. Hence, in order to show the failure of the amalgamation property in a semilinear variety with the congruence extension property, it suffices to find a V-formation whose algebras are totally ordered, and that does not have a one-sided amalgam in residuated chains. We do exactly this, and we exhibit a V-formation, which we call $\mathscr{V}\mathscr{S}$-formation, given by 2-potent commutative integral residuated chains that does not have a one-sided amalgam in the class of totally ordered residuated lattices. This entails that, if V is a variety of semilinear residuated lattices with the congruence extension property, and such that the algebras in the $\mathscr{V}\mathscr{S}$-formation belong to V, then V does not have the amalgamation property. In particular we get the following results.

Theorem 1 ([7])**.** *The following varieties do not have the amalgamation property:*

1. *MTL-algebras;*
2. *Semilinear commutative residuated lattices;*
3. *Semilinear commutative integral residuated lattices;*
4. *Semilinear commutative* FL*-algebras;*
5. *n-potent MTL-algebras for $n \geqslant 2$.*

The result about semilinear commutative residuated lattices has recently been shown in [5]. Using some algebraic constructions (*rotations* and *liftings*) we are also able to adapt our counterexample to construct a V-formation consisting of, respectively, involutive and pseudocomplemented FL-algebras; thus in particular we obtain the following:

Theorem 2 ([7])**.** *The following varieties do not have the amalgamation property:*

1. *IMTL-algebras;*
2. *SMTL-algebras;*
3. *n-potent IMTL and SMTL-algebras for $n \geqslant 2$.*

We observe that given the previously mentioned bridge theorem, our results entail that the logics corresponding to the varieties in Theorems 1 and 2 do not have the deductive interpolation property.

Finally, we mention that the algebras involved in the V-formation that yields the counterexample can be constructed by means of a new construction that we introduce in order to be able to construct new chains from known ones. Such construction extends and generalizes the *partial gluing construction* in [6], and allows us to find other countably many varieties of residuated lattices without the amalgamation property.

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Justifying Homotopical Logic Two Ways

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The talk will be divided into three parts. The first one will be dedicated to the notion of internal language in category theory and its foundational significance. The second part will use the model-theoretic viewpoint to justify the logical status of the Univalence Axiom. Finally, the rest of the talk will be dedicated to connecting both of the approaches together in order to justify the idea behind homotopical logic.

Certain categories with rich enough structure (like various kinds of topoi) can be used as a model of a “mathematical universe”. One can differentiate the use of the internal language of the category in question, i.e.’reasoning within the universe’, with ’meta-theoretical’ or external reasoning (which is more similar to the ordinary style of mathematical reasoning), that is,’reasoning about the universe.’

This opens a possibility for various conceptually interesting interactions. Proving statements internally allows for the generalization of obtained results to a certain extent. However, I will give a few examples where external and internal presentations of the same concept do not coincide and why this distinction is important for modern practice regarding foundations of mathematics, such as univalent foundations.

As one can use set-theoretic constructions to encode all mathematics inside the hierarchy of sets, the same can supposedly be done with certain categories (e.g., (inf,1)- topoi). More importantly, if one uses intuitionistic principles internally, but the meta-theoretical is classical, the proof as a whole can’t be considered constructive. As noted by T. Coquand [7], initial work on the simplicial model of univalent foundations used classical meta-theory, which resulted in non-constructive proofs. Thus, internal reasoning can prevent making unwarranted statements. Furthermore, it allows us to use an appropriately defined internal language as a tool to internalize principles of meta-theoretical nature, such as homotopy-invariance, shaping the idea behind homotopical logic (borrowing the name from A. Joyal’s talk [2]).

The Univalence Axiom states that $=_U (A, B) \cong (A \cong B)$. The axiom was first introduced by Voevodsky and was motivated by the idea of homotopy theory, i.e., everything is considered under "homotopy equivalence". Another way to read the axiom is captured in S. Awodey’s slogan for mathematical structuralism that "isomorphic structures can be identified".

In the homotopical interpretation of Martin-Löf’s type theory, types are interpreted as spaces, i.e., they are essentially homotopy types of spaces. The introduction of the Axiom of Univalence into type theory allows to give a formal status to the intuition that any object is invariant under the homotopy equivalence, i.e. it internalizes the aforementioned external meta-theoretical principle.

The general idea goes as follows. Given a first-order language, one can formulate a list of axioms in a given fragment of logic $\mathbb{T}$. $\mathbb{T}$, then, is modeled by some mathematical objects. For example, having ZFC as a theory, it is modeled by the von Neumann universe **V**, which is constructed using the meta-theory. In particular, **V** is constructed using set theory as a semi-informal meta-language and, consequently, ZFC will be the object language. From the foundational point of view, models (or in this case, **V**) live in some set-theoretic universe, which is the case in the set-theoretical model theory.

From the perspective of category-theoretic semantics, this just means that "traditional" set-theoretical models "live" in the category **Sets**, i.e., semantics is a functor from some category that represents syntax of a given theory to the universe **Sets**. Instead of **Sets**, we can consider something with more structure, obtaining other kinds of models.

The addition of homotopical interpretation to type theory is essentially about interpreting identity as a homotopy equivalence. If two types are homotopy equivalent, they are identical, i.e., there exists an identity type between them. Homotopy theory is a study of objects invariant under continuous deformation, which presupposes a weaker notion of identity.

Finally, one can see that it makes perfect sense that the universe corresponding to the homotopical meta-theory above is not **Sets** and set-theory. In particular, for the intensional dependent type theory with the Univalence Axiom, correct models can be categorically described as infinity topoi. HoTT, as a theory, is the presupposed internal language of these categories. The fact that the Univalence Axiom holds only in "homotopical" models, makes the shift from set-theoretical foundations to category theory foundationally justified.

On the one hand, the example above gives a definitive positive answer to the question why one needs internal language in order to distinguish between different meta-theoretical principles since the presupposed internal language of the univalent universes is not set-theoretic.

On the other hand, the categories for which HoTT are supposed to be the internal language are characterized by weaker invariance criteria. This poses an interesting question about the logical status of the Univalence Axiom and the justification for homotopical logic.

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Modelling concepts with affordance relations

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Abstract

We aim to formalize the Gibsonian notion of an *affordance* relation and use it to explain (i) actions as labeled ternary relations arising from the interactions between actors and objects in a context (environment), and (ii) relational concepts as abstractions arising from actions.

The framework

We intend to model affordances and actions in the framework of property and information systems in the sense of [7] and [6]. A *property system* (*P-system*) is a structure $\langle U, V, f\rangle$, where U is a non-empty set whose elements are called *objects*, V is a set whose elements are called *properties*, and $f\colon U \to 2^V$ is a mapping called an *information function*; we do not require that $f(x) \neq \varnothing$. A statement $a \in f(u)$ can be interpreted as "Object u possesses property a". If U is finite, then a property system is definitionally equivalent to a formal dyadic context of Wille [8], by observing that the function $f\colon U \to 2^V$ can be replaced by a relation $R_f \subseteq U \times V$, where $u\ R_f\ a$ if and only if $a \in f(u)$ which has the same informational content.[1]

If we think of a property system as describing possible states of an attribute—such as "color" or "language spoken"—we extend it by the definition of an aggregate structure: An *attribute system* (*A-system*) [7] is a structure $\mathscr{S} := \langle U, \Omega, \{V_a : a \in \Omega\}, f\rangle$ where

1. U is a non-empty set of objects,

[1] At this stage of our investigation we suppose that we have a correct description of the world, i.e. what we observe is true.

2. Ω is a set of property labels or attributes, and V_a is a set of possible values of $a \in \Omega$,
3. $f: U \times \Omega \to \bigcup_{a\in\Omega} 2^{V_a}$ is a choice function, where $f(x,a) \subseteq V_a$. Equivalently, we may define $f : U \to \prod_{a\in\Omega} 2^{V_a}$.

So, if $a \in \Omega$ is a property label *weight*, then V_{weight} may be a set of rational numbers in some interval that can serve as numerical expressions of the weight of an object (e.g., in kilograms or pounds), or any value that makes sense. It could also be some aggregated value such as "low", "medium", "high" etc.

We call $\langle U, \Omega, \{V_a : a \in \Omega\}\rangle$ the *skeleton* of $\mathscr{S}$. The product $U \times \prod_{a\in\Omega} 2^{V_a}$ collects all possible vectors of value sets that can be associated with some element of U. An information function now picks one element from $\prod_{a\in\Omega} 2^{V_a}$ for each $x \in U$.

An element $x \in U$ is called *deterministic*, if $|f(x,a)| \leqslant 1$ for all $a \in \Omega$ or every projection of the vector attribute to x is either an empty set or a singleton subset of V_a. The set of all deterministic elements of $\mathscr{S}$ is denoted by $D_{\mathscr{S}}$. The characterization stems from the fact that the role of the choice function is to narrow down the possibilities for the values of a with respect to the object x. If $|f(x,a)| = 1$, then we know the exact value of the property a for x, or if $|f(x,a)| = \varnothing$, then we know that x does not have this property at all. This is why we call a system with $|f(x,a)| \leqslant 1$ deterministic. If $|f(x,a)| \geqslant 2$ the set has different possibilities of interpretation, see [3]. If U is finite and $U = D_{\mathscr{S}}$, then $\mathscr{S}$ is called an *information system* (in the sense of 6).

Operationalizing affordances

A direction on operationalization of affordances was suggested in [2]:

> A formalization of affordance relations needs to provide crisp and fuzzy structures, mechanisms for spatial and temporal change, as well as contextual modeling.

The basic setup of an affordance relation consists of a set U of an agent's abilities, a set E of features of the environment, and a binary relation $R \subseteq U \times E$. Chemero [1, p. 189] writes

> Affordances [...] are relations between the abilities of organisms and features of the environment. Affordances, that is, have the structure Affords–φ (feature,ability).

We expand this notion by regarding an affordance in a first step as a relation $\varphi \subseteq A \times O \times E$ between actors, objects and properties of the environment, where $\varphi(a,o,e)$ is interpreted as

> Entity o affords action $\texttt{Act}_\varphi$ to the actor (or perceiver, agent) a in the environment (context) e.

The initial notion of an affordance is quite coarse, and all three components require further description. Therefore, we extend the concept as follows: Suppose that for a set A of actors, a set O of entities or objects, and a set E of environmental factors we have deterministic information systems

$$\mathscr{I}_A = \left\langle A, \Omega_A, \{V_q^A : q \in \Omega_A\}, f_A : A \to \textstyle\prod_{q\in\Omega_A} V_q^A \right\rangle,$$
$$\mathscr{I}_O = \left\langle O, \Omega_O, \{V_q^O : q \in \Omega_O\}, f_O : O \to \textstyle\prod_{q\in\Omega_A} V_q^O \right\rangle,$$
$$\mathscr{I}_E = \left\langle E, \Omega_E, \{V_q^A : q \in \Omega_E\}, f_E : E \to \textstyle\prod_{q\in\Omega_A} V_q^E \right\rangle.$$

Each of these information systems is interpreted as a description, respectively, of actors, entities, or the environment. We now define an *affordance* as a relation

$$\varphi \subseteq \{\langle a, f_A(a)\rangle : a \in A\} \times \{\langle o, f_O(o)\rangle : o \in O\} \times \{\langle e, f_E(e)\rangle : e \in E\}.$$

Thus, an affordance is a ternary relation that holds among actors with properties, objects with properties, and environments (contexts) with properties. See Figure 1 for a pictorial interpretation.

Figure 1: The triples of vectors of the same color constitute the affordance φ and its corresponding action $\texttt{Act}_\varphi$. We identify, respectively, actors, objects, and environments that cannot be distinguished by available properties.

INFORMATION SYSTEM FOR ACTORS

$A/_\sim$ \ Ω_A	p_1	p_2	p_3	p_4	p_5	p_6
$[a_1]$	$v_{a_1}^{p_1}$	$v_{a_1}^{p_2}$	$v_{a_2}^{p_3}$	$v_{a_2}^{p_4}$	$v_{a_2}^{p_5}$	$v_{a_2}^{p_6}$
$[a_2]$	$v_{a_2}^{p_1}$	$v_{a_2}^{p_2}$	$v_{a_2}^{p_3}$	$v_{a_2}^{p_4}$	$v_{a_2}^{p_5}$	$v_{a_2}^{p_6}$
$[a_3]$	$v_{a_3}^{p_1}$	$v_{a_3}^{p_2}$	$v_{a_3}^{p_3}$	$v_{a_3}^{p_4}$	$v_{a_3}^{p_5}$	$v_{a_3}^{p_6}$
$[a_4]$	$v_{a_4}^{p_1}$	$v_{a_4}^{p_2}$	$v_{a_4}^{p_3}$	$v_{a_4}^{p_4}$	$v_{a_4}^{p_5}$	$v_{a_4}^{p_6}$
$[a_5]$	$v_{a_5}^{p_1}$	$v_{a_5}^{p_2}$	$v_{a_5}^{p_3}$	$v_{a_5}^{p_4}$	$v_{a_5}^{p_5}$	$v_{a_5}^{p_6}$
$[a_6]$	$v_{a_6}^{p_1}$	$v_{a_6}^{p_2}$	$v_{a_6}^{p_3}$	$v_{a_6}^{p_4}$	$v_{a_6}^{p_5}$	$v_{a_6}^{p_6}$
$[a_7]$	$v_{a_7}^{p_1}$	$v_{a_7}^{p_2}$	$v_{a_7}^{p_3}$	$v_{a_7}^{p_4}$	$v_{a_7}^{p_5}$	$v_{a_7}^{p_6}$

$\times$

INFORMATION SYSTEM FOR ENTITIES

$O/_\sim$ \ Ω_O	q_1	q_2	q_3	q_4	q_5	q_6	q_7
$[o_1]$	$v_{o_1}^{q_1}$	$v_{o_1}^{q_2}$	$v_{o_1}^{q_3}$	$v_{o_1}^{q_4}$	$v_{o_1}^{q_5}$	$v_{o_1}^{q_6}$	$v_{o_1}^{q_7}$
$[o_2]$	$v_{o_2}^{q_1}$	$v_{o_2}^{q_2}$	$v_{o_2}^{q_3}$	$v_{o_2}^{q_4}$	$v_{o_2}^{q_5}$	$v_{o_2}^{q_6}$	$v_{o_2}^{q_7}$
$[o_3]$	$v_{o_3}^{q_1}$	$v_{o_3}^{q_2}$	$v_{o_3}^{q_3}$	$v_{o_3}^{q_4}$	$v_{o_3}^{q_5}$	$v_{o_3}^{q_6}$	$v_{o_3}^{q_7}$
$[o_4]$	$v_{o_4}^{q_1}$	$v_{o_4}^{q_2}$	$v_{o_4}^{q_3}$	$v_{o_4}^{q_4}$	$v_{o_4}^{q_5}$	$v_{o_4}^{q_6}$	$v_{o_4}^{q_7}$
$[o_5]$	$v_{o_5}^{q_1}$	$v_{o_5}^{q_2}$	$v_{o_5}^{q_3}$	$v_{o_5}^{q_4}$	$v_{o_5}^{q_5}$	$v_{o_5}^{q_6}$	$v_{o_5}^{q_7}$
$[o_6]$	$v_{o_6}^{q_1}$	$v_{o_6}^{q_2}$	$v_{o_6}^{q_3}$	$v_{o_6}^{q_4}$	$v_{o_6}^{q_5}$	$v_{o_6}^{q_6}$	$v_{o_6}^{q_7}$
$[o_7]$	$v_{o_7}^{q_1}$	$v_{o_7}^{q_2}$	$v_{o_7}^{q_3}$	$v_{o_7}^{q_4}$	$v_{o_7}^{q_5}$	$v_{o_7}^{q_6}$	$v_{o_7}^{q_7}$
$[o_8]$	$v_{o_8}^{q_1}$	$v_{o_8}^{q_2}$	$v_{o_8}^{q_3}$	$v_{o_8}^{q_4}$	$v_{o_8}^{q_5}$	$v_{o_8}^{q_6}$	$v_{o_8}^{q_7}$

$\times$

INFORMATION SYSTEM FOR ENVIRONMENTS

$E/_\sim$ \ Ω_E	r_1	r_2	r_3	r_4	r_5	r_6	r_7
$[e_1]$	$v_{e_1}^{r_1}$	$v_{e_1}^{r_2}$	$v_{e_1}^{r_3}$	$v_{e_1}^{r_4}$	$v_{e_1}^{r_5}$	$v_{e_1}^{r_6}$	$v_{e_1}^{r_7}$
$[e_2]$	$v_{e_2}^{r_1}$	$v_{e_2}^{r_2}$	$v_{e_2}^{r_3}$	$v_{e_2}^{r_4}$	$v_{e_2}^{r_5}$	$v_{e_2}^{r_6}$	$v_{e_2}^{r_7}$
$[e_3]$	$v_{e_3}^{r_1}$	$v_{e_3}^{r_2}$	$v_{e_3}^{r_3}$	$v_{e_3}^{r_4}$	$v_{e_3}^{r_5}$	$v_{e_3}^{r_6}$	$v_{e_3}^{r_7}$
$[e_4]$	$v_{e_4}^{r_1}$	$v_{e_4}^{r_2}$	$v_{e_4}^{r_3}$	$v_{e_4}^{r_4}$	$v_{e_4}^{r_5}$	$v_{e_4}^{r_6}$	$v_{e_4}^{r_7}$
$[e_5]$	$v_{e_5}^{r_1}$	$v_{e_5}^{r_2}$	$v_{e_5}^{r_3}$	$v_{e_5}^{r_4}$	$v_{e_5}^{r_5}$	$v_{e_5}^{r_6}$	$v_{e_5}^{r_7}$
$[e_6]$	$v_{e_6}^{r_1}$	$v_{e_6}^{r_2}$	$v_{e_6}^{r_3}$	$v_{e_6}^{r_4}$	$v_{e_6}^{r_5}$	$v_{e_6}^{r_6}$	$v_{e_6}^{r_7}$
$[e_7]$	$v_{e_7}^{r_1}$	$v_{e_7}^{r_2}$	$v_{e_7}^{r_3}$	$v_{e_7}^{r_4}$	$v_{e_7}^{r_5}$	$v_{e_7}^{r_6}$	$v_{e_7}^{r_7}$
$[e_8]$	$v_{e_8}^{r_1}$	$v_{e_8}^{r_2}$	$v_{e_8}^{r_3}$	$v_{e_8}^{r_4}$	$v_{e_8}^{r_5}$	$v_{e_8}^{r_6}$	$v_{e_8}^{r_7}$
$[e_9]$	$v_{e_9}^{r_1}$	$v_{e_9}^{r_2}$	$v_{e_9}^{r_3}$	$v_{e_9}^{r_4}$	$v_{e_9}^{r_5}$	$v_{e_9}^{r_6}$	$v_{e_9}^{r_7}$

$\varphi = \{$  $\}$

Actions and concepts from affordances

A tuple in an affordance φ is interpreted as "action $\texttt{Act}_\varphi$ is afforded for actor a by entity o in the context e". A tuple $\langle \mathscr{I}_A, \mathscr{I}_O, \mathscr{I}_E, \varphi\rangle$ is called an *affordance structure*. For example, if actors are cleaning robots, objects are charging stations, and environments are interiors (e.g., offices or apartments), then we may think about, e.g., cyan triples from φ as

> charging stations of the type $[o_2]$, afford docking of robots of the type $[a_2]$ in interiors of types $[e_3]$ and $[e_7]$,

and similarly about triples of the two remaining colors.

Let us suppose that we also have a set of labels to tag affordances. These may be seen as finite strings of symbols over a finite alphabet Σ (that is, the Kleene closure Σ^* of Σ). Let us suppose that Σ is the standard Latin alphabet. Then, we may attribute the label '`dock`' to the affordance φ and thus obtain an action $\texttt{dock}_\varphi$ of *docking*. Thus actions are labeled affordances, more formally, they are elements of the set $\Sigma^* \times \texttt{Aff}$, where `Aff` is a set of affordances.

There is a similarity between $\texttt{dock}_\varphi$ and the action $\texttt{dock}_\psi$ of docking a ship in a shipyard. Clearly, we have different information systems composed of ships as actors, landing piers as objects, and shipyards as environments. Still further, we can see the similarity to the action $\texttt{dock}_\zeta$ where the information systems for the affordance ζ concern spaceships (actors), space stations (objects), and a low-gravity environment. We can now define concepts as abstractions from all actions of the type $\texttt{dock}_\delta$, where δ is an affordance. Speaking formally, the concept `DOCK` is a set of all actions of docking

$$\texttt{DOCK} := \{\texttt{dock}_\varphi \mid \varphi \text{ is an affordance}\}\,.$$

We may say that concepts are abstractions from all those affordances to which we tend to attribute the same element of Σ^*.

The purpose of this presentation is to give details of our constructions and relate them to the other well-known formal theories of concepts, for example [8] and [5].

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The weak Robinson property

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The connection between the algebraic property of amalgamation and the syntactic property of interpolation has received considerable attention in the literature in the frameworks of model theory [1], abstract algebraic logic [2], universal algebra [5], and residuated structures [4, 6]. Explicitly, if a logic $\vdash$ is algebraized by a variety $\mathscr{V}$ that has the congruence extension property, then $\mathscr{V}$ has the amalgamation property if and only if $\vdash$ has the deductive interpolation property. This "bridge theorem" provides a powerful technique for establishing the deductive interpolation property via the amalgamation property, and vice versa. However, for varieties that lack the congruence extension property, failure of the amalgamation property does not necessarily imply failure of the deductive interpolation property. A natural problem is therefore to describe a property that, when combined with the deductive interpolation property, is equivalent to the amalgamation property. In this work, we identify such a property and provide an algebraic characterization, offering a potential pathway to resolving certain open problems in the area.

Amalgamation and interpolation

The term "amalgamation" refers to the process by which two algebras are combined while preserving a common subalgebra. To express this notion formally, let $\mathscr{K}$ be a class of similar algebras. A *doubly injective span* in $\mathscr{K}$ is a 5-tuple $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \varphi_B, \varphi_C \rangle$, consisting of algebras $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathscr{K}$ and embeddings $\varphi_B \colon \boldsymbol{A} \to \boldsymbol{B}$ and $\varphi_C \colon \boldsymbol{A} \to \boldsymbol{C}$. The class $\mathscr{K}$ is said to have the *amalgamation property* (for short, AP) if for every doubly injective span $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \varphi_B, \varphi_C \rangle$ in $\mathscr{K}$, there exist an algebra $\boldsymbol{D} \in \mathscr{K}$, and embeddings $\psi_B \colon \boldsymbol{B} \to \boldsymbol{D}$ and $\psi_C \colon \boldsymbol{C} \to \boldsymbol{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.

The AP for a variety can be characterized in terms of its free algebras, which can be then reflected in a property of the corresponding equational consequence relation of the variety. In particular, we may focus on the equational consequence relation for a fixed countably infinite set of variables X. Formally, the consequence relation $\vDash_{\mathscr{K}}$ on the set of equations $Eq(X)$ (pairs of formulas over X) is defined as follows for $\Sigma \cup \{\varepsilon\} \subseteq Eq(X)$:

$$\Sigma \vDash_{\mathscr{V}} \varepsilon :\Longleftrightarrow \text{ for any homomorphism } h \text{ from the formula algebra over } X \text{ to some } \boldsymbol{A} \in \mathscr{K}, \\ \Sigma \subseteq \ker(h) \Longrightarrow \varepsilon \in \ker(h).$$

Given $\Sigma \cup \Gamma \subseteq Eq(X)$, we write $\Sigma \vDash_{\mathscr{V}} \Gamma$ if $\Sigma \vDash_{\mathscr{V}} \gamma$ for all $\gamma \in \Gamma$, and denote by $Var(\Gamma)$ the set of variables occurring in Γ. If $\mathscr{V}$ is a variety, then the equational consequence relation

$\vDash_{\mathscr{V}}$ is finitary. Moreover, if $\vdash$ is an algebraizable logic with equivalent algebraic semantics $\mathscr{V}$, then there are mutually inverse translations between $\vdash$ and $\vDash_{\mathscr{V}}$.

A variety $\mathscr{V}$ has the *Robinson property* (for short, RP) if for any $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(X)$ such that $Var(\Sigma) \cap Var(\Pi) \neq \varnothing$ and $Var(\{\varepsilon\}) \cap Var(\Pi) \subseteq Var(\Sigma)$, whenever

(i) $\Sigma \vDash_{\mathscr{V}} \delta \Longleftrightarrow \Pi \vDash_{\mathscr{V}} \delta$ for all $\delta \in Eq(X)$ with $Var(\delta) \subseteq Var(\Sigma) \cap Var(\Pi)$;

(ii) $\Sigma \cup \Pi \vDash_{\mathscr{V}} \varepsilon$,

then $\Sigma \vDash_{\mathscr{V}} \varepsilon$.

Theorem 1 (cf. [5, Thm. 13])**.** *A variety has the amalgamation property if and only if it has the Robinson property.*

The RP (and hence the AP) implies the deductive interpolation property, whose algebraic counterpart is the "generalized amalgamation property with injections", introduced by Kihara and Ono in [4]. Formally, a variety $\mathscr{V}$ is said to have the *deductive interpolation property* (for short, DIP) if for any $\Sigma \cup \{\varepsilon\} \subseteq Eq(X)$ such that $Var(\Sigma) \cap Var(\{\varepsilon\}) \neq \varnothing$, whenever

(i) $\Sigma \vDash_{\mathscr{V}} \varepsilon$,

then there exists $\Delta \subseteq Eq(X)$ with $Var(\Delta) \subseteq Var(\Sigma) \cap Var(\{\varepsilon\})$ such that

(ii) $\Sigma \vDash_{\mathscr{V}} \Delta$;

(iii) $\Delta \vDash_{\mathscr{V}} \varepsilon$.

Conversely, the DIP implies the AP in the presence of the congruence extension property. Recall that a variety $\mathscr{V}$ has the *congruence extension property* (for short, CEP) if for every $\boldsymbol{D} \in \mathscr{V}$, subalgebra $\boldsymbol{C}$ of $\boldsymbol{D}$, and congruence Θ of $\boldsymbol{C}$, there exists a congruence Φ of $\boldsymbol{D}$ such that $\Theta = \Phi \cap C^2$.

The syntactic counterpart of the CEP is the extension property (see [6, Sec. 8.2]). A variety $\mathscr{V}$ is said to have the *extension property* (for short, EP) if for any $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(X)$, whenever

(i) $\Sigma \cup \Pi \vDash_{\mathscr{V}} \varepsilon$,

then there exists $\Delta \subseteq Eq(X)$ with $Var(\Delta) \subseteq Var(\Pi \cup \{\varepsilon\})$ such that

(ii) $\Sigma \vDash_{\mathscr{V}} \Delta$;

(iii) $\Delta \cup \Pi \vDash_{\mathscr{V}} \varepsilon$.

On the syntactic side, the connection between the RP and DIP can be made explicit in the form of the following theorem, which appeared first in [3].

Theorem 2 (cf. [5, Thm. 22]). (a) *If a variety has the Robinson property, then it has the deductive interpolation property.*

(b) *If a variety has the deductive interpolation property and the extension property, then it has the Robinson property.*

The weak Robinson property

Theorem 2 naturally poses the challenge of defining a property weaker than the EP that, when combined with the DIP, is equivalent to the RP (and hence also the AP). To address this challenge, we define the weak Robinson property.

We say that a variety $\mathscr{V}$ has the *weak Robinson property* (for short, WRP) if for any $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(X)$ such that $Var(\Sigma) \cap Var(\Pi) \neq \varnothing$ and $Var(\{\varepsilon\}) \cap Var(\Pi) \subseteq Var(\Sigma)$, whenever

(i) $\Sigma \vDash_{\mathscr{V}} \delta \Longleftrightarrow \Pi \vDash_{\mathscr{V}} \delta$ for all $\delta \in Eq(X)$ with $Var(\delta) \subseteq Var(\Sigma) \cap Var(\Pi)$;

(ii) $\Sigma \cup \Pi \vDash_{\mathscr{V}} \varepsilon$;

(iii) $\Pi \vDash_{\mathscr{V}} \rho \implies \Sigma \vDash_{\mathscr{V}} \rho$ for all $\rho \in Eq(X)$ with $Var(\rho) \subseteq Var(\Sigma)$,

then $\Sigma \vDash_{\mathscr{V}} \varepsilon$.

This property provides a positive answer to the challenge posed above. By definition, it is immediate that the RP implies the WRP. Also, it can be shown that the WRP is implied by the EP, and indeed is strictly weaker than the EP, since, e.g., the variety of groups has the RP but not the EP. The following theorem states that the conjunction of the WRP and DIP are in fact equivalent to the RP.

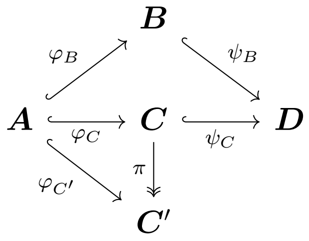
Theorem 3. *A variety has the Robinson property if and only if it has the weak Robinson property and the deductive interpolation property.*

This theorem implies that there are varieties that lack the WRP but still satisfy the DIP—as observed in [7], the variety of semigroups has the DIP despite failing the AP.

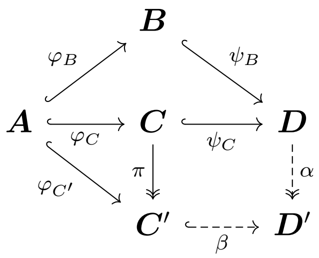
In parallel to the connection between the EP and CEP, there exists an algebraic counterpart of the WRP. We say that a variety $\mathscr{V}$ has the *weak congruence extension property* (for short, WCEP) if for any algebra $\boldsymbol{D} \in \mathscr{V}$ with subalgebras $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathscr{V}$ such that $\boldsymbol{A}$ is a common subalgebra of $\boldsymbol{B}$ and $\boldsymbol{C}$, and $\boldsymbol{D}$ with the inclusion maps $\boldsymbol{B} \hookrightarrow \boldsymbol{D}$, $\boldsymbol{C} \hookrightarrow \boldsymbol{D}$ is the pushout of the inclusion maps $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$, $\boldsymbol{A} \hookrightarrow \boldsymbol{C}$, the following holds: for every congruence Θ of $\boldsymbol{C}$ such that $\Theta \cap A^2$ is the least congruence Δ_A of $\boldsymbol{A}$, there exists a congruence Φ of $\boldsymbol{D}$ such that $\Theta = \Phi \cap C^2$.

A categorical approach provides a natural way to characterize properties such as the CEP using diagrams (see, e.g., [1]). Similarly, we can give a categorical description of the

WRP. We say that a variety $\mathscr{V}$ has the *weak extension property* (for short, WEP) if for each commuting diagram in $\mathscr{V}$ of the form



where $\varphi_{C'}$ is injective, π is surjective and $\boldsymbol{D}$ with the embeddings $\psi_B\colon \boldsymbol{B} \to \boldsymbol{D}$ and $\psi_C\colon \boldsymbol{C} \to \boldsymbol{D}$ is the pushout of the doubly injective span $\langle \boldsymbol{A}, \varphi_B, \varphi_C\rangle$, there exist a surjective homomorphism $\alpha\colon \boldsymbol{D} \to \boldsymbol{D}'$ and an embedding $\beta\colon \boldsymbol{C}' \to \boldsymbol{D}'$, such that the following diagram commutes:



Theorem 4. *Let $\mathscr{V}$ be a variety. Then the following are equivalent:*

(1) *$\mathscr{V}$ has the weak Robinson property.*

(2) *$\mathscr{V}$ has the weak congruence extension property.*

(3) *$\mathscr{V}$ has the weak extension property.*

Concluding remarks

Despite the progress made, there remain several intriguing gaps in our understanding. Crucially, we do not yet have any example of a variety that has the WRP but lacks the AP and CEP. Even if these properties are distinct in the setting of universal algebra, it would be interesting to identify families of varieties that have or do not have the WRP as a method either for refuting the RP or for showing that it is equivalent to the DIP. For example, it is known that the variety of lattice-ordered groups does not have the RP, but the question of whether it has the DIP is open; by showing that this variety has the WRP, the question would be answered negatively.

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Semilinear Finitary Extensions of Pointed Abelian logic

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In this talk, we will consider extensions of pointed abelian logic determined by subquasivarieties of a class of pointed Abelian ℓ-groups. In particular, we will focus on those quasivarieties which are generated by chains.

Łukasiewicz logic in its infinitely-valued version was introduced by Łukasiewicz and Tarski [30] in 1930 and since then it was proved to be one of the most prominent non-classical logics. This logic is by itself a member of the family of many-valued logics often used to model some aspects of vagueness. Also, it has deep connections with other areas of mathematics such as continuous model theory, error-correcting codes, geometry, algebraic probability theory, etc. [4, 13, 24, 27].

Abelian logic is a well-known (finitary) contraclassical paraconsistent logic. This logic was independently introduced by Meyer and Slaney [26] and by Casari [3] and it is also called the logic of Abelian ℓ-groups [2] or Abelian Group Logic [28]. This terminology follows from the fact that the matrix models of Abelian logic consist of Abelian ℓ-groups and their positive cones as filters of designated elements (there is also a version of Abelian logic in which the only designated element is the neutral element of the group, which will not be considered here).

Pointed Abelian logic is expansion of Abelian logic, where we add a new constant symbol **f** to the language, but do not add any axioms. This new constant greatly improves the expressive power of our logic. In particular, this logic contains several important extensions, the most important of which was Łukasiewicz unbound logic (see[6]).

The varieties of MV-algebras, classified by Komori in [23], correspond to varieties of positively ($f \geqslant 0$) pointed abelian ℓ-groups, as shown by Young in [29] via the Mundici functor (for the definition of the Mundici functor, see [4]). We will generalize this to classification of all pointed ℓ-groups.

The whole variety of pointed Abelian ℓ-groups is generated by $\{\mathbf{R}_{-1}, \mathbf{R}_1\}$ and $\{\mathbf{Q}_{-1}, \mathbf{Q}_1\}$, respectively. The subvarieties of $\mathbf{HSP}(\mathbf{R}_1)$ are determined by algebras $\mathbf{Z}_n$ and $\mathbf{Z}_n \ltimes \mathbf{Z}_0$ for $n, > 0$ and subvarieties of $\mathbf{HSP}(\mathbf{R}_{-1})$ are determined by algebras $\mathbf{Z}_n$ and $\mathbf{Z}_n \ltimes \mathbf{Z}_0$ for $n, < 0$. This can be generalized to the description of all subvarieties of pointed Abelian ℓ-groups.

Our next goal is to generalize this classification to all quasivarieties generated by chains. The motivation for this approach is that these quasivarieties correspond to semilinear extensions of pointed abelian logic. In [15] Gispert described all semilinear finite extensions of Łukasiewicz logic by describing all universal classes of MV-chains, thus (using [8]) giving a classification of all quasi-varieties of MV-algebras generated by chains. We show that this classification can also be applied to pointed ℓ-groups by proving the following lemma.

Lemma 1. *Let $\boldsymbol{A}$ be an Abelian ℓ-group a let $\boldsymbol{B}$ be a convex subgroup of $\boldsymbol{A}$, and let be $b \in B$ a strong unit in $\boldsymbol{B}$. Then* $\mathbf{ISPP_U}(\boldsymbol{A}_b) = \mathbf{ISPP_U}(\boldsymbol{B}_b)$.

In other words, this lemma tells us that we can restrict ourselves to groups with a strong unit. These are known to be equivalent to MV-algebras via the Mundici functor. Therefore, we can describe all quasivarieties of pointed Abelian ℓ-groups generated by chains as it is stated in the following theorem.

Theorem 1. *Let* $\mathbf{S}$ *denote any finitely generated dense ℓ-subgroup of* $\mathbf{R}$ *such that* $\mathbf{S} \cap \mathbf{Q} = \mathbf{Z}$. *Every subquasivariety of pAb generated by chains is equal to*

$$\mathbf{ISPP_U}(\{\mathbf{Z_n} \mid \mathbf{n} \in \mathbf{A}\} \cup \{\mathbf{Z_n} \ltimes \mathbf{Z_m} \mid \mathbf{n} \in \mathbf{B}, \mathbf{m} \in \gamma(\mathbf{n}) \cup \{\mathbf{S_d} \mid \mathbf{d} \in \mathbf{C}\}\}),$$

for some $A, B, C \subseteq Z$, and $\gamma : n \mapsto \gamma(n) \subseteq \mathrm{div}(n)$, where $\mathrm{div(n)}$ *stands for the set of all divisors of $n \in Z$.*

Although the above result can be derived quite easily from [15] using our Lemma 1 and the Mundici functor, we try to prove these results without using theory of MV-algebras. We believe that this will lead to a significant simplification of the proofs used. In the last section we give an axiomatization of these quasivarieties.

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Amalgamation in classes of involutive commutative residuated lattices

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Introduction

Amalgamation is explored in this talk within classes of involutive commutative residuated lattices that are non-divisible, non-integral, and non-idempotent. Several classes of algebras significant to us are designated by a distinctive notation:

$\mathfrak{A}^{\mathfrak{c}}$ the class of abelian o-groups
$\mathfrak{I}$ the class of involutive FL_e-algebras
$\mathfrak{S}$ the class of odd or even idempotent-symmetric involutive FL_e-algebras

Adjunct to $\mathfrak{I}$,

- the superscript $\mathfrak{c}$ means restriction to totally-ordered algebras,
- the superscript $\mathfrak{sl}$ means restriction to semilinear algebras,
- the subscript $\mathfrak{o}$ means restriction to odd algebras,
- the subscript $\mathfrak{e}$ means restriction to even algebras,
- the subscript $\mathfrak{e}_{\mathfrak{i}}$ means restriction to even algebras having an idempotent falsum constant,
- the subscript $\mathfrak{e}_{\mathfrak{n}}$ means restriction to even algebras having a non-idempotent falsum constant,

When multiple letters appear in the subscript, they denote the union of the corresponding classes. For instance $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{o}\mathfrak{e}_{\mathfrak{i}}}$ refers to the class of idempotent-symmetric involutive FL_e-chains which are either odd or even with an idempotent falsum constant.

First we delve into the Amalgamation Property within subclasses of $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{o}\mathfrak{e}}$. We show that several subclasses of these structures fail to satisfy the Amalgamation Property (Theorem 1), including the classes of odd and even ones. This failure stems from the same underlying reason as in the case of discrete linearly ordered abelian groups with positive normal homomorphisms [3]. Conversely, it is proven that three subclasses of them exclusively comprising algebras that are idempotent-symmetric possess the Amalgamation Property (Theorem 2), albeit fail the Strong Amalgamation Property (Theorem 3). The failure of the Strong Amalgamation Property in these subclasses can be attributed to the same underlying reason observed in the class of linearly ordered abelian groups with positive homomorphisms [1].

Then we shift our focus from these classes of chains to the semilinear varieties of FL_e-algebras that they generate. Our goal is to transfer the Amalgamation Property, or its failure, from the specific classes of chains to the generated varieties. We conclude that every variety of semilinear involutive commutative (pointed) residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property (Theorem 4). This result strengthens a recent proof by W. Fussner and S. Santschi, which established that the variety of semilinear involutive commutative residuated lattices lacks the Amalgamation Property [2, Theorem 5.2]. Furthermore, we demonstrate that the varieties of idempotent-symmetric, semilinear, odd involutive residuated lattices, as well as idempotent-symmetric, semilinear, odd or even involutive residuated lattices, exhibit the Transferable Injections Property (Theorem 5), a strengthening of the Amalgamation Property.

Amalgamation in classes of $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{oe}}$

Theorem 1. *The classes $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{e}}$, $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{e}_{\mathfrak{i}}}$, $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{e}_{\mathfrak{n}}}$, along with every class of involutive FL_e-chains which contains $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{o}}$, fail the Amalgamation Property.*

Theorem 2. *The classes $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{o}}$, $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{e}}$, and $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{oe}}$ each satisfy the Amalgamation Property.*

Theorem 3. *The classes $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{o}}$, $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{e}}$, and $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{oe}}$ do not satisfy the Strong Amalgamation Property.*

Amalgamation in the generated semilinear varieties

Theorem 4. *Every variety of semilinear involutive commutative (pointed) residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property.*

Theorem 5. *The varieties $\mathfrak{S}^{\mathfrak{sl}}_{\mathfrak{o}}$ and $V(\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{e}})$ have the Transferable Injections Property.*

Techniques

The core principle of our approach relies on leveraging the intrinsic layer group decomposition of the algebras in $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{oc}}$ [4] and an associated categorical equivalence [5]. This strategic direct system decomposition facilitates the independent execution of amalgamation within each distinct layer. Subsequently, these layer-wise amalgams are leveraged to construct the overall amalgam of the algebras via the functor detailed in [5] (see Fig. 1).

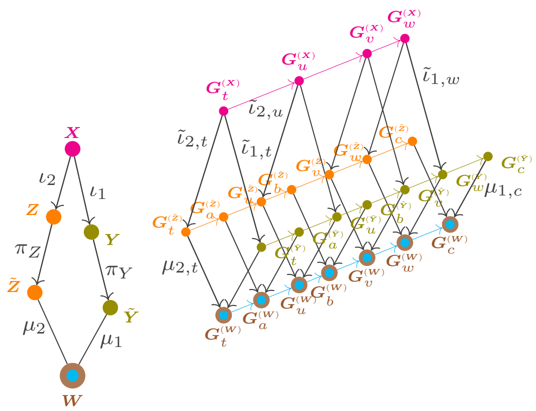

Figure 1: Brief visual illustration of the main constructions: "Layerwise" amalgamation in $\mathfrak{A}^{\mathfrak{c}}$ (right), and the corresponding amalgamation in $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{oc}}$ (left).

As an example, proving Theorem 6 was necessary to convert the cyan direct system into the brown one. Additionally, several techniques for embedding direct systems into those over larger index sets were developed to construct the embeddings shown in Fig. 1.

Theorem 6. *For any direct system $\langle \mathbf{L}_u, \varsigma_{u\to v}\rangle_\kappa$ of torsion-free partially ordered abelian groups over an arbitrary chain κ, there exists a direct system $\langle \widehat{\mathbf{G}}_u, \varsigma_{u\to v}\rangle_\kappa$ of abelian o-groups. In this system the abelian group reducts of the $\mathbf{L}_u$'s and the transitions remain unchanged, while, for every $u \in \kappa$, the ordering relation of $\widehat{\mathbf{G}}_u$ is an extension of the ordering relation of $\mathbf{L}_u$.*

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The Beth companion: making implicit operations explicit. Part I

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Let K be a class of algebras.

Definition 1. An n-ary partial function on K is a tuple $\langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$, where each $f^{\mathbf{A}}$ is a function $f^{\mathbf{A}} \colon X \to A$ for some $X \subseteq A^n$. The set X is then called the *domain* of $f^{\mathbf{A}}$ and denoted with $\mathsf{dom}(f^{\mathbf{A}})$.

We are interested in particular partial functions that exhibit a behaviour similar to that of term functions.

Definition 2. Let $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$ be a partial function on K. Then f is *preserved by homomorphisms* when for every homomorphism $h \colon \mathbf{A} \to \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n\rangle \in \mathsf{dom}(f^{\mathbf{A}})$ we have $\langle h(a_1), \ldots, h(a_n)\rangle \in \mathsf{dom}(f^{\mathbf{B}})$ and

$$h(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)).$$

Definition 3. A first-order formula φ in the language of K is said to be *functional* in K when for every $\mathbf{A} \in \mathsf{K}$ and $a_1, \ldots, a_n \in A$ there exists at most one $b \in A$ such that $\mathbf{A} \vDash \varphi(a_1, \ldots, a_n, b)$.

A functional formula φ induces an n-ary partial function $\varphi^{\mathbf{A}}$ on each $\mathbf{A} \in \mathsf{K}$ with domain

$$\mathsf{dom}(\varphi^{\mathbf{A}}) = \{\langle a_1, \ldots, a_n\rangle \in A^n : \text{there exists } b \in A \text{ such that } \mathbf{A} \vDash \varphi(a_1, \ldots, a_n, b)\}$$

defined for every $\langle a_1, \ldots, a_n\rangle \in \mathsf{dom}(\varphi^{\mathbf{A}})$ as $\varphi^{\mathbf{A}}(a_1, \ldots, a_n) = b$, where b is the unique element of A such that $\mathbf{A} \vDash \varphi(a_1, \ldots, a_n, b)$.

Definition 4. A partial function $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$ on K is called *implicit* if there exists a functional formula φ such that $f^{\mathbf{A}} = \varphi^{\mathbf{A}}$ for each $\mathbf{A} \in \mathsf{K}$. In this case we say that φ defines f.

Definition 5. An *implicit operation* of K is an implicit partial function of K that, moreover, is preserved by homomorphisms.

Example 1. Let DL be the variety of bounded distributive lattices and φ the formula

$$(x \wedge y \approx 0) \,\&\, (x \vee y \approx 1).$$

If $\mathbf{A} \in \mathsf{DL}$, then $\mathbf{A} \vDash \varphi(a, b)$ if and only if b is a complement of a in $\mathbf{A}$. Since complements in bounded distributive lattices are unique when they exist and are preserved by bounded lattice homomorphisms, we conclude that φ defines an implicit operation of DL.

It is well known that the formulas preserved by homomorphisms are precisely the *existential positive formulas*; that is, the formulas built from equations and $\perp$ using only existential quantifiers, conjunctions, and disjunctions. As a consequence, we obtain the following characterization of implicit operations.

Proposition 1. *Let* K *be an elementary class and* f *a partial function on* K*. Then* f *is an implicit operation if and only if it is defined by an existential positive formula.*

Definition 6. An n-ary implicit operation f is said to be *interpolated in* K by a set $\mathscr{T}$ of n-ary terms in the language of K when for every $\mathbf{A} \in \mathsf{K}$ and $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\mathbf{A}})$ there exists $t \in \mathscr{T}$ such that

$$f^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n).$$

Intuitively, the implicit operation f is made explicit by the terms in $\mathscr{T}$.

Example 2. Every term-function is clearly an implicit operation that is interpolated by a single term. However, not all implicit operations can be interpolated by terms. For instance, there is no set of terms in the language of bounded distributive lattices interpolating the operation of taking complements in DL.

Definition 7. A class of algebras is said to have the *strong Beth definability property* when each of its implicit operations can be interpolated by a set of terms.

The name "strong Beth definability property" is motivated by the resemblance to the *Beth definability property* in logic. When a finitary logic is algebraized by a quasivariety K, the former has the Beth definability property iff all epimorphisms in K are surjective [1, Thm. 3.12] (see also [2]). For quasivarities the strong Beth definability property can be conveniently phrased in the following equivalent form, where we recall that a *primitive positive formula* (*pp formula*, for short) is a conjunction of equations prenexed by existential quantifiers.

Proposition 2. *A quasivariety has the* strong Beth definability property *if and only if each of its implicit operations defined by pp formulas can be interpolated by a single term.*

The strong Beth definability property corresponds to a condition stronger than the surjectivity of epimorphisms (see [3, 4] for this correspondence in the case of modal and intuitionistic logic), which is defined as follows:

Definition 8. A class of algebras K has the *strong epimorphism surjectivity property (SES property*, for short*)* when for every homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ and $b \in B - f[A]$ there exists a pair of homomorphisms $g, h\colon \mathbf{B} \to \mathbf{C}$ with $\mathbf{C} \in \mathsf{K}$ such that $g \circ f = h \circ f$ and $g(b) \neq h(b)$.

Theorem 1. *A universal class has the SES property iff it has the strong Beth definability property.*

When K is a quasivariety, the SES property can be formulated in purely categorical terms: K has the SES property iff all monomorphisms in K are regular.

Our main contribution is a method to optimally expand a given class of algebras into one where all implicit operations can be made explicit. The guiding example is the variety of Boolean algebras, which can be obtained by adding the operation of taking complements to the variety of bounded distributive lattices.

Let $\mathscr{F}$ be a set of implicit operations of K. We denote by $\mathcal{L}_{\mathscr{F}}$ the language obtained by expanding the language of K with function symbols acting as names for the operations in $\mathscr{F}$. Whenever $\mathbf{A} \in \mathsf{K}$ has the property that $f^{\mathbf{A}}$ is a total function for each $f \in \mathscr{F}$, we can expand $\mathbf{A}$ to an algebra $\mathbf{A}[\mathcal{L}_{\mathscr{F}}]$ in the language $\mathcal{L}_{\mathscr{F}}$ by interpreting the new function symbols with the respective implicit operations in $\mathscr{F}$. We can then consider the collection of such expansions:

$$\mathsf{K}[\mathcal{L}_{\mathscr{F}}] := \{\mathbf{A}[\mathcal{L}_{\mathscr{F}}] : \mathbf{A} \in \mathsf{K} \text{ and } f^{\mathbf{A}} \text{ is total for each } f \in \mathscr{F}\}.$$

Definition 9. An implicit operation f of K is said to be *extendable* in K if every $\mathbf{A} \in \mathsf{K}$ can be embedded into some $\mathbf{B} \in \mathsf{K}$ such that $f^{\mathbf{B}}$ is a total function.

Definition 10. A *pp expansion* M of K is the class of subalgebras of $\mathsf{K}[\mathcal{L}_{\mathscr{F}}]$ for some set $\mathscr{F}$ of extendable implicit operations of K that are defined by pp formulas. If moreover M has the strong Beth definability property, we call it a *Beth companion* of K.

Example 3.

(i) The variety of Boolean algebras is a Beth companion of the variety of bounded distributive lattices.

(ii) The variety of implicative semilattices is a Beth companion of the variety of Hilbert algebras.

(iii) The variety of Heyting algebras of depth at most 2 is a Beth companion of the variety of pseudocomplemented distributive lattices.

(iv) The variety of Abelian groups is a Beth companion of the quasivariety of cancellative commutative monoids.

But not every class of algebras has a Beth companion.

Example 4.

(i) The variety of (commutative) monoids does not admit a Beth companion.

(ii) Infinitely many varieties of Heyting algebras do not admit a Beth companion. In particular, for every $n \geqslant 5$ the variety generated by the n-element Heyting chain lacks a Beth companion.

As a main result, we obtain that Beth companions have the following desirable properties:

Theorem 2.

(i) *A Beth companion of a quasivariety is a quasivariety.*

(ii) *All the Beth companions of a quasivariety are term-equivalent.*

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Exchangeability and statistical models in non-classical logic

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Probability theory and fuzzy logic are both theories used when one aims at performing some sort of inference in uncertain or imprecise situations. These two theories have been combined in many different ways and for this talk we recall [1], where the authors define the algebraic counterpart of a random variable within a conservative expansion of Łukasiewicz logic. The probabilistic setting used there is the one of subjective probability, as introduced by Bruno de Finetti. Another point of view on probability theory is the so-called frequentist approach. In this case, the probability of an event is defined by the frequency of that event based on previous observations. In this setting, it is common to define a statistical model that fits the observed data, and to derive the properties of the hidden probability distribution in this way. Perhaps surprisingly, subjective probability is related to frequentist probability via de Finetti's work, and in particular via the notion of *exchangeability* that, loosely speaking, shows how statistical models appear in a Bayesian framework, and how probabilities can come from statistics.

In this talk, we mainly discuss the state of the art on the search for a *good* notion of exchangeability and statistical models in non-classical logic, staring with the results of [6]. In particular, we discuss the notion of exchangeability in the setting of Łukasiewicz logic, give a version of the celebrated de Finetti's theorem in algebraic logic, and present the definition and some results on statistical models that are new to the LATD audience.

Classically, exchangeability is defined as follows. Let T be a non-empty set with a σ-algebra of subsets $\mathscr{X}$ and let $\overline{\mathscr{X}}$ be the smallest σ-algebra on T^ω that contains all sets of type $C(E_{i_1},\ldots,E_{i_k}) = \prod_n E_n$ with $E_n = T$ for $n \notin \{i_1,\ldots,i_k\}$. A generic measure σ on $\overline{\mathscr{X}}$ is called *exchangeable* if for any permutation π of $\{i_1,\ldots,i_k\}$,

$$\sigma\left(C(E_{i_1},\ldots,E_{i_k})\right) = \sigma\left(C(E_{\pi(i_1)},\ldots,E_{\pi(i_k)})\right).$$

For any measure μ on $\mathscr{X}$, let $\overline{\mu}$ denote the unique product measure defined on $\overline{\mathscr{X}}$ using μ. A generic measure σ on $\overline{\mathscr{X}}$ is called *presentable* if there exists a measure ν on the set $\mathcal{P}$ of all probability measure on $\mathscr{X}$ such that for any $A \in \overline{\mathscr{X}}$,

$$\sigma(A) = \int_{\mathcal{P}} \overline{\mu}(A) d\nu(\mu).$$

Then, de Finetti's theorem (in the version of Hewitt and Savage [5]) gives sufficient conditions for the two notions to coincide.

To give a non-classical version of this result, the algebraic framework used will be the one of MV-algebras. This is mainly due to the fact that an algebraic theory of probability is encoded in MV-algebras by the notion of *states*. Via a suitable version of the Riesz representation theorem, the so-called Kroupa-Panti theorem, states of MV-algebras are in one-one correspondence with probability measures on a σ-algebra that depends on A.

In particular, we will work with a subclass of the infinitary variety $\mathbf{RMV}_\sigma$ of σ-complete Riesz MV-algebras, that is, MV-algebras closed to multiplication by elements of $[0,1]$ and to countable suprema, see [4]. The subclass needed is the pre-variety generated by $[0,1]$. Algebras in $ISP([0,1])$ are called σ-semisimple and they can be characterized as follows, when they are countably generated.

For a countable cardinal κ, $\mathsf{Borel}([0,1]^\kappa)$ denotes the MV-algebra of $[0,1]$-valued and Borel-measurable functions over the domain $[0,1]^\kappa$. Such algebra is proved to be the free κ-generated algebra in $\mathbf{RMV}_\sigma$ in [2]. If needed, when X is a countable set of generators, we will write $\mathsf{Borel}(X)$ instead of $\mathsf{Borel}([0,1]^{|X|})$. For a topological space T, $\mathscr{BO}(T)$ denotes the σ-algebra of its Borel subsets, that is, the σ-algebra generated by open subsets of T. Now, with a characterization proved in [2], a (at most) countably generated algebra A is σ-semisimple iff there exist sets $\kappa \leq \omega$ and V, with V being an arbitrary intersection of Borel subsets of $[0,1]^\kappa$, such that $A \simeq \mathsf{Borel}([0,1]^\kappa)|_V$, the algebra of restrictions to V of elements of $\mathsf{Borel}([0,1]^\kappa)$.

In this setting, we define an appropriate counterpart of a product measure. To do so, thinking of states as an algebraic dual of probability measures, we will first give a characterization for the coproduct of objects in $\mathbf{RMV}_\sigma$.

Proposition 1. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of σ-semisimple algebras in $\mathbf{RMV}_\sigma$. For any $n \in \mathbb{N}$, let $A_n \simeq \mathsf{Borel}(X_n)|_{V_n}$, where we assume the sets X_n to be countable and pairwise disjoint and we assume each V_n to be a Baire subset of $[0,1]^{X_n}$. Then the free product $\bigoplus_n A_n$ exists in $\mathbf{RMV}_\sigma$ and*

$$\bigoplus_n A_n = \mathsf{Borel}\left(\bigcup_n X_n\right)|_V \qquad \text{with } V = \prod_{n\in\mathbb{N}} V_n.$$

Using this characterization of coproduct, we will define the notion of *presentable state* and *exchangeable state*, and prove that they coincide on any $\mathsf{Borel}([0,1]^\kappa)$, when κ is countable.

Definition 1. Let $A \in \mathbf{RMV}_\sigma$ countably generated, $A \simeq \mathsf{Borel}([0,1]^\kappa)|_V$. Take the coproduct $\bigoplus_\omega A$ of A with itself countably many times. A σ-state $s : \bigoplus_\omega A \to [0,1]$ is called *weakly exchangeable* if the associate measure (in the sense sketched above, by the Kroupa-Panti theorem), on the product of countable copies of $(V, \mathscr{BO}(V))$, is exchangeable in the classical sense. Similarly, the σ-state s is called *weakly presentable* if the associated measure is presentable in the classical sense.

Theorem 1 (Weak de Finetti's exchangeability). *Let κ be a countable cardinal. A state on $\mathsf{Borel}([0,1]^\kappa)$ is weakly exchangeable if, and only if, it is weakly presentable.*

In the second part of the talk we present a logico-algebraic take on the notion of a *statistical model* introduced in [6] and further discussed in [3].

Formally, for $\kappa \leq \omega$, a logico-algebraic statistical model is a function $\eta = (\eta_i)_{i\in\kappa} \colon P \to \Delta_\kappa$, where $P \subseteq [0,1]^d$ is an intersection of Borel measurable sets and Δ_κ is the standard κ-dimensional simplex. When $\kappa = \omega$, we take Δ_ω to be $\{x \in [0,1]^\omega \mid \sum_{i=1}^{\infty} x_i \leq 1\}$, which is known to be closed (and convex) and therefore it is a Borel subset of $[0,1]^\omega$. The intuition behind this definition is the following:

- $[0,1]^\kappa$ is the set of observations on the real world and $\mathsf{Borel}([0,1]^\kappa)$ is the algebra of many-valued events;
- the set $P \subseteq [0,1]^d$ is the set of states of the world, or parameters, we allow d to be any countable cardinal;
- the tuple of functions $\eta := (\eta_i)_{i\in\kappa} \colon P \to [0,1]^\kappa$ is our statistical model: to each parameter $\mathbf{x} \in P$ it associates the tuple $(\eta_i(\mathbf{x}))_{i\in\kappa}$. Each $\eta_i \colon [0,1]^d \to [0,1]$ is a Borel measurable function.

We call κ-dimensional any statistical model whose codomain is Δ_κ.

We will show how κ-dimensional statistical models can be interpreted in the category of σ-complete Riesz MV-algebras and how they can be seen as a suitable pre-sheaf, whose domain is a category of *parameters*.

Lastly, if time permits, we present a (work in progress) approach to exchangeability for states that does not require the Kroupa-Panti theorem and it is related to the more general framework of subjective decision theory.

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Tropicalization through the lens of Łukasiewicz logic, with a topos theoretic perspective

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The main aim of this paper is to show that the topics of Łukasiewicz logic, semirings and tropical structures fruitfully meet, by combining the ideas of [1], [6], [15] and [4]. This gives rise to a topos theoretic perspective to Łukasiewicz logic, see [2]. Our aim seems to be completely in line with the spirit of the following remark from [12]:

"*The confluence of geometric, combinatorial and*

logical-algebraic techniques on a common problem

is one of the manifestations of the unity of mathematics."

Tropicalization, originally, is a method aimed at simplifying algebraic geometry and it is used in many applications. It can be seen as a part of idempotent algebraic geometry. Tropical geometry produces objects which are combinatorially similar to algebraic varieties, but piecewise linear. That is, tropicalization attaches a polyhedral complex to an algebraic variety, obtaining a kind of algebraic geometry over idempotent semifields.

Originally the idea was focused on complex algebraic geometry and started from a field (or ring) K and its polynomials, an idempotent semiring $(S, \wedge, +, 0, 1)$ and a valuation $v : K \to S$. A semiring is an abelian monoid $(A, +, 0)$ with a multiplicative monoid structure $(A, \cdot, 1)$ satisfying the distributive laws and such that $a \cdot 0 = 0 \cdot a = 0$ for all $a \in A$. A semiring is idempotent if $+$ is idempotent.

Usually $S = G \cup \{\infty\}$ where G is a totally ordered abelian group and ∞ is an infinite extra element, but we suggest to be more liberally inspired by [15]: first, G can be any lattice ordered abelian group, and $S = G \cup \{\infty\}$ is a semifield (the *Rump semifield* of G) . Moreover, we point out that the passage from G to S and conversely is functorial, even an equivalence. So we propose here a *functorial tropicalization*.

A polynomial p with coefficients in K is replaced by a polynomial $Trop(p)$ with coefficients in S. In the tropicalization $Trop(p)$ of p, the polynomial product of K becomes $+$ in S, the polynomial sum in K becomes the infimum in S. The constants $k \in K$ become their value $v(k)$. The idea is that the tropicalized polynomial $Trop(p)$ should be simpler than p but retain the combinatorial structure of p.

In this paper we intend to contribute to what we mean by tropical mathematics, which for us, in a very broad sense, is the use of idempotent semirings in mathematics.

The most used semiring in tropicalization is the tropical semiring

$$\mathbb{R}^{maxplus} = (\mathbb{R} \cup \{-\infty\}, max, +, -\infty, 0)$$

(or its dual with min instead of max). It is used in many contexts, from algebraic geometry to optimization to phylogenesis. An application to differential equations is given in [8]. The variety generated by $\mathbb{R}^{maxplus}$ includes the semiring reduct of $[0,1]$, but not the other way round. Instead, if we consider the negative cone of the reals, $\mathbb{R}^{neg} = (\mathbb{R}^{\leqslant 0} \cup \{-\infty\}, max, +, -\infty, 0)$, we have that the varieties generated by $[0,1]$ and $\mathbb{R}^{neg}$ are the same. Another famous semiring is the tropical semiring $\mathbb{N}^{trop} = (\mathbb{N} \cup \{+\infty\}, min, +, +\infty, 0)$. Once again, the varieties generated by $\mathbb{N}^{trop}$ and $[0,1]$ are the same. However, $\mathbb{N}^{trop}$ and $[0,1]$ have different first order theories. Thus, looking at these two varieties from a logical point of view, some differences emerge. This justifies approaching tropical algebraic structures from a logical point of view.

One can try to apply the same tropicalization idea to more general frameworks, like universal algebraic geometry by Plotkin [13]. This paper proposes a possible generalization of tropicalization of algebraic structures using many valued logic and especially fuzzy logic. In fact, the algebraic structures of fuzzy Łukasiewicz logic (MV-algebras) have a natural structure of idempotent semiring. Note that MV-semirings are mentioned in [7] in the context of weighted logics.

Łukasiewicz logic is a fuzzy logic, that is a logic where the set of values is the real interval $[0,1]$, rather than the case $\{0,1\}$ of classical logic. The connectives of Łukasiewicz logic are $x \oplus y = min(x+y, 1)$ (replacing OR) and $\neg x = 1 - x$ (replacing NOT). Then AND gets replaced by $x \odot y = max(0, x+y-1)$ (the Łukasiewicz product). We have the tertium non datur, $x \oplus \neg x = 1$, the non contradiction law $x \odot \neg x = 0$, but not the idempotency: $x \oplus x \neq x$. Despite the lack of idempotency, MV-algebras are deeply connected with idempotent structures.

Łukasiewicz logic can be axiomatized as follows, where $x \to y$ means $\neg x \oplus y$:

1. $x \to (y \to x)$
2. $(x \to y) \to ((y \to z) \to (x \to z))$
3. $((x \to y) \to y)) \to ((y \to x) \to x)$
4. $(\neg x \to \neg y) \to (y \to x)$.

The only rule is modus ponens: from $x \to y$ and x derive y.

Like the semantic counterpart of classical logic is given by Boolean algebras, the semantic counterpart of Łukasiewicz logic is given by MV-algebras. These are algebraic structures generalizing Boolean algebras and widely used in many applications, from quantum mechanics to games to functional analysis. The interplay between semirings and MV-algebras is very interesting. Every MV-algebra is in a natural way (better, in two natural dual ways) a semiring, as explained in [5], [4].

Namely, MV-algebras are structures $(A, \oplus, 0, 1, \neg)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg 0 = 1$, $x \oplus 1 = 1$, $\neg\neg x = x$ and $\neg(\neg x \oplus y) \oplus y) = \neg(\neg y \oplus x) \oplus x)$.

MV-algebras are also lattices under the ordering $x \leqslant y$ such that there is z with $y = x \oplus z$. We let also $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \ominus y = x \odot \neg y = \neg(\neg x \oplus y)$.

The main example of MV-algebra is $[0, 1]$ with $x \oplus y = min(x + y, 1)$ and $\neg x = 1 - x$. By [3] it follows that $[0, 1]$ generates the variety of MV-algebras.

The semiring reducts of an MV-algebra A are $(A, 0, 1, \wedge, \oplus)$ and $(A, 0, 1, \vee, \odot)$. They are isomorphic semirings and an isomorphism is given by the negation (see [4]).

In order to define other interesting MV-algebras it is convienient to introduce Mundici functor Γ, see [9]. Γ is a functorial equivalence between MV-algebras and Abelian ℓ-groups with strong unit. This equivalence was originally used in order to clarify the relations between AF C^*-algebras and their K_0 group. $\Gamma(G, u)$ is the interval $[0, u]$ of G where $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$.

Another important example is the Chang MV-algebra $C = \Gamma(Z \overset{lex}{\times} Z, (1, 0))$ where $\overset{lex}{\times}$ denotes lexicographic product of groups. We call $V(C)$ the variety generated by C and $V(C)$-algebras the elements of $V(C)$. The algebra C has a very special role in MV-algebras theory. $\Gamma^{-1}(C)$ is the K_0 of Behncke-Leptin C^*-algebra $A_{0,1}$, see [10].

An equivalence related to Γ is the one called Δ between abelian ℓ-groups and perfect MV-algebras, see ([6]), that is, MV-algebras generated by their infinitesimals. Namely, if G is an abelian ℓ-group, then $\Delta(G) = \Gamma(Z \overset{lex}{\times} G, (1, 0))$. So, C can also be defined as $\Delta(Z)$.

We can think of the logic of perfect MV-algebras as the logic of the quasi true or quasi false.

We focus on algebraic models of an equational extension of Łukasiewicz logic. Actually we consider the extension of Łukasiewicz logic given by the equation in the section 3.1.

It is often stressed that enriched valuations represent "perturbations" of valuations of classical models. In our case we can speak of "infinitesimal perturbation" of classical propositional logic, that in turn, means infinitesimal perturbation of Boolean algebras. Although it seems rather exotic to consider infinitesimally perturbed Boolean algebras as models of a propositional logic [11], it happens that these models can have an elegant algebraic characterization. Indeed, we can see that, albeit by means of a categorical equivalence, such models can be described by a Stone space and a family of lattice ordered abelian groups. Actually we will consider weak boolean products of lattice ordered abelian groups. It will be quite interesting that such a logic may have theories interpreted into points of the presheaf topos over the multiplicative monoid of integers.

As future work, we note that we propose a tropicalization functor that should be compared with [2] and [14]. In [2], theorem 2.1, the authors relate subgroups of the rationals to the points of a natural topos, and in this paper we transfer this correspondence into a result on a category of countable, perfect, linearly ordered MV algebras, via the functor Δ. It should be possible to extend this result to more general categories of MV algebras, using other topoi. In [14] in particular the problem of negation is addressed, which is an issue also for us, since the functor θ gives structures which do not have a natural involutive negation

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On Non-Classical Polyadic Algebras

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Algebraization of logic has been widely studied by logicians ever since G. Boole discovered the connection between classical propositional logic and two-element Boolean-type algebras. Afterwards, A. Mostowski, A. Tarski, and P. Halmos developed the lattice-based [8], cylindric [4], and polyadic [9] algebraization of classical quantified logic, respectively. To further generalize these ideas, researchers have explored the algebraization of nonclassical quantified logics, leading to the development of structures such as polyadic MV-algebras [5], polyadic BL-algebras [6], polyadic Rasiowa-implicative algebras [2] and cylindric Heyting algebras [10].

Following this line of research, we first define polyadic algebras over algebraically-implicative logics [1]. After constructing functional polyadic algebras, we prove the functional representation theorem, which encompasses many known results for non-classical polyadic algebras.

Let's first fix some notations. Give two sets I, J with $J \subseteq I$. We call a mapping $\sigma : I \to I$ a transformation of I and denote the identity transformation by ι. For $\sigma, \tau \in I^I$, $\sigma J \tau$ means that $\sigma(i) = \tau(i)$ for all $i \in J$. That is, σ and τ agrees on J. Also, we denote $\sigma(I \setminus J)\tau$ as $\sigma J_* \tau$, i.e. σ and τ agree on the complement of I. If $\sigma J_* \iota$, we say J *supports* σ.

Let $\mathscr{L}_{\forall\exists} = \langle \mathscr{O}, \forall, \exists, \mathbf{P}, \mathbf{F}, Var, \rho \rangle$ be a first-order language where $\{\to\} \subseteq \mathscr{O}$ is a set of propositional connectives, $\mathbf{P}(\mathbf{F})$ is a set of relation (functional) symbols, Var is a set of variables, and $\rho : \mathscr{O} \to \omega$ is an arity function.

Similar to classical polyadic algebra developed by Halmos in [9], we first define polyadic $\langle \mathscr{L}_{\forall\exists}, I \rangle$- algebra $\mathbf{A}$ is as

$$\langle A, (\circ^{\mathbf{A}} : \circ \in \mathscr{O}), \forall^{\mathbf{A}}, \exists^{\mathbf{A}}, S^{\mathbf{A}} \rangle$$

where $\circ^{\mathbf{A}} : A^n \to A$ if $\rho(\circ) = n$, $\forall^{\mathbf{A}}, \exists^{\mathbf{A}} : \mathscr{P}_\omega(I) \to A^A$, and $S^{\mathbf{A}} : I^I \to A^A$ such that the following axioms are satisfied :

- $S^{\mathbf{A}}_\iota x = x$;
- $S^{\mathbf{A}}_\sigma(S^{\mathbf{A}}_\tau x) = S^{\mathbf{A}}_{\sigma\tau} x$, for all $\sigma, \tau \in I^I$;
- $S^{\mathbf{A}}_\sigma(\circ^{\mathbf{A}}(x_1, \ldots, x_{\rho(\circ)})) = \circ^{\mathbf{A}}(S^{\mathbf{A}}_\sigma x_1, \ldots, S^{\mathbf{A}}_\sigma x_n)$, for all $\circ \in \mathscr{O}$, $\sigma \in I^I$;
- $S^{\mathbf{A}}_\sigma Q^{\mathbf{A}}_J x = S^{\mathbf{A}}_\tau Q^{\mathbf{A}}_J x$ for all $Q \in \{\forall, \exists\}$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that $\sigma J_* \tau$;

- $Q_J^{\mathbf{A}} S_\sigma^{\mathbf{A}} x = S_\sigma^{\mathbf{A}} Q_{\sigma^{-1}(J)}^{\mathbf{A}} x$ for all $Q \in \{\forall, \exists\}$, $J \subseteq_\omega I$, and $\sigma, \tau \in I^I$ such that σ is injective on $\sigma^{-1}(J)$.

We then denote L as algebraically-implicative predicate logic with the language $\mathscr{L}_{\forall\exists}$ as in [1]. By lemma 2.9.11 in [1], $\mathbf{A}$ is an algebra of truth values for L, or an L-algebra, if there is a set of equations $\mathscr{E}$ such that the following quasi-equations hold in $\mathbf{A}$ for each $\alpha \approx \beta \in \mathscr{E}$:

- $\alpha(\varphi) \approx \beta(\varphi)$, for each axiom φ of L
- $\bigwedge \mathscr{E}[\Gamma] \Rightarrow \alpha(\varphi) \approx \beta(\varphi)$ for each rule $\Gamma \vdash_{\mathrm{L}} \varphi$ of L
- $\bigwedge \mathscr{E}[x \leftrightarrow y] \Rightarrow x \approx y$

Then we define that a polyadic $\langle \mathscr{L}_{\forall\exists}, I\rangle$-algebra $\mathbf{A}$ is called a polyadic L-algebra if it satisfies the following equations and quasi-equations :

- Axioms of L-algebras;
- Axioms (T1)-(T8) for all $\sigma \in I^I$ and $J \subseteq_\omega I$ as in [2].

On the other hands, following the definition in [2], we say a value $\mathscr{L}_{\forall\exists}$-algebra $\mathbf{V}$ is an algebra of the form

$$\langle V, (\circ^{\mathbf{V}} : \circ \in \mathscr{O}), \forall^{\mathbf{V}}, \exists^{\mathbf{V}}\rangle$$

where $\circ^{\mathbf{V}} : V^{\rho(\circ)} \to V$ is a $\rho(\circ)$-ary operation on V for each $\circ \in \mathscr{O}$, and $Q^{\mathbf{V}} : \mathscr{P}(V) \rightharpoonup V$ is a partial unary second-order operation with domain on power set $\mathscr{P}(V)$ of V for each $Q \in \{\forall, \exists\}$.

Therefore, given a value $\mathscr{L}_{\forall\exists}$-algebra $\mathbf{V}$ and two sets X, I. A *functional polyadic* $\langle \mathscr{L}, I\rangle$-*algebra* $\bar{\mathbf{V}}$ is of the form

$$\langle V^{X^I}, (\circ^{\bar{\mathbf{V}}} : \circ \in \mathscr{O}), \forall^{\bar{\mathbf{V}}}, \exists^{\bar{\mathbf{V}}}, S^{\bar{\mathbf{V}}}\rangle$$

where $\circ^{\bar{\mathbf{V}}} : (V^{X^I})^{\rho(\circ)} \to V^{X^I}$, $\forall^{\bar{\mathbf{V}}}, \exists^{\bar{\mathbf{V}}} : \mathscr{P}_\omega(I) \to [V^{X^I}, V^{X^I}]$, and $S^{\bar{\mathbf{V}}} : I^I \to End(\mathbf{V})$ are defined as follows :

- $(\circ^{\bar{\mathbf{V}}}(p_1, \dots, p_{\rho(\circ)}))(\vec{x}) = \circ^{\mathbf{V}}(p_1(\vec{x}), \dots, p_{\rho(\circ)}(\vec{x}))$ for all $p_1, \dots, p_{\rho(\circ)} \in V^{X^I}$ and $\vec{x} \in X^I$;
- $(\forall_J^{\bar{\mathbf{V}}} p)(\vec{x}) = \forall^{\mathbf{V}}\Big(\{p(\vec{y}) : \vec{x} J_* \vec{y}\}\Big)$,
  for all $p \in V^{X^I}$, $J \subseteq_\omega I$, and $\vec{x}, \vec{y} \in X^I$; similarly for $\exists^{\bar{\mathbf{V}}}$.
- $(S_\sigma^{\bar{\mathbf{V}}} p)(\vec{x}) = p(\sigma_* \vec{x})$, where $(\sigma_* \vec{x})_i = (\vec{x})_{\sigma(i)}$
  for all $\sigma \in I^I$ and $\vec{x} \in X^I$.

Note that we use $[V^{X^I}, V^{X^I}]$ to denote that $\forall_J^{\bar{\mathbf{V}}} p$ and $\exists_J^{\bar{\mathbf{V}}}$are total functions from X^I to $\mathbf{V}$. If $\langle V, (\circ^{\mathbf{V}} : \circ \in \mathscr{O})\rangle \in \mathbf{ALG}^*(\mathrm{L})$, the algebra of reduced models of L, and $\forall^{\mathbf{V}}$ and $\exists^{\mathbf{V}}$ are respectively the generalized meet and join operations, then we say $\bar{\mathbf{V}}$ is a functional polyadic L-algebra. We can prove a similar theorem as in [9] :

Theorem 1. *Every functional polyadic* L*-algebra is a polyadic* L*-algebra.*

To see the connection with algebraically-implicative predicate logic, let $\mathfrak{M}$ be a reduced model for L and $P_{\mathfrak{M}}$ is the interpretation of predicate symbols $P \in \mathbf{P}$ in $\mathfrak{M}$. We can show the following lemma.

Lemma 1. *Let $\mathscr{F}(\mathfrak{M})$ be a subalgebra of $\bar{\mathbf{A}}$ (with $X = M$ and $I = Var$) generated by $\{P_{\mathfrak{M}} \mid P \in \mathbf{P}\}$. Then $\mathscr{F}(\mathfrak{M})$ is a functional polyadic $\langle \mathscr{L}_{\forall\exists}, Var\rangle$-algebra.*

To prove the converse case, it's similar to the classical case that we need to impose some further constrain on the polyadic algebras. We say an element a of a polyadic $\langle \mathscr{L}_{\forall\exists}, I\rangle$-algebra has a finite support $J \subseteq I$ if $S_\sigma a = S_\tau a$ for all $\sigma, \tau \in I^I$ such that $\sigma J \tau$. A polyadic $\langle \mathscr{L}_{\forall\exists}, I\rangle$-algebra is locally finite if every element has a finite support. Hence, we can prove the following functional representation theorem.

Theorem 2. *Every locally finite polyadic* L*-algebra of infinite dimension is isomorphic to a functional polyadic* L*-algebra.*

As a case study, we investigate the algebraization of first-order relevant logics. Let $\mathscr{L}_{RQ} = \langle \{\wedge, \vee, \sim, \circ, 1, \to\}, Con, Pred, \forall, I, \rho\rangle$ where Con is a set of name constant symbols (i.e. 0-ary functional symbols), $Pred$ is a set of predicate symbols of varying arities, I is a countable set of variables, and ρ is an arity function. A *polyadic $\langle \mathscr{L}_{RQ}, I\rangle$-De Morgan Monoid* is an algebra of the form:

$$A := \langle A; \wedge, \vee, \sim, \circ, \to, 1, \langle \forall_J^A \mid J \subseteq_\omega I\rangle, \langle S_\sigma^A \mid \sigma \in I^{(I)}\rangle\rangle$$

that satisfies the following axioms:

(Poly) Axioms of polyadic $\langle \mathscr{L}_\forall, I\rangle$-algebras

(DMM) The defining equations of De Morgan Monoids

(Q1) $\forall_J 1 = 1$;

(Q2) $\forall_J p \leqslant p$;

(Q3) $\forall_J (p \wedge q) = \forall_J p \wedge \forall_J q$;

(Q4) $\forall_J \forall_J p = \forall_J p = \sim \forall_J \sim \forall_J p$;

(Q5) $\forall_J (p \to q) \leqslant (\forall_J p \to \forall_J q)$;

(Q6) $\forall_J (\forall_J p \to \forall_J q) = \forall_J p \to \forall_J q$;

(Q7) $\forall_J (p \vee q) = \sim \forall_J \sim p \vee \forall_J q$.

We can construct functional polyadic De Morgan Monoids similarly. Therefore, we have the following theorem.

Theorem 3. *Every functional polyadic $\langle \mathscr{L}_{RQ}, I\rangle$-De Morgan Monoid is a polyadic $\langle \mathscr{L}_{RQ}, Var\rangle$-De Morgan Monoid.*

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Game semantics for weak depth-bounded approximations to classical propositional logic

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Tractable deductive systems approximating classical propositional logic (CPL) have interest in areas that require models of bounded rationality (see [3]). In a series of papers culminating in [1], *depth-bounded* approximations have been studied, which can be intuitively related to the deduction power of resource-bounded agents. Among these approximations, the so-called *weak* ones are defined in terms of the *depth* of derivations within a KE-KI system, and are decidable in polynomial time whenever their associated depth is suitably parameterized. The 0-depth approximation can't be characterized by a set of finitely valued matrices. So far, two alternative semantics have been given to characterize that basic approximation and, recursively, all of its successors. Namely, a *modular* one and a 3-valued *non-deterministic* one [1, Sec. 1.3, 1.5]. Both are well motivated and intuitive for the 0-depth approximation, though not necessarily for those of greater depth. In this work, we introduce a game semantics which in our opinion provides a more intuitive framework for the whole hierarchy of approximations. Namely, we define a game where *negative* constraints are associated with understanding the informational meaning of the connectives, while resource consumption is transparently modeled by the *expense* of questions that are within a finite number. Although related to standard dialogical accounts [4], our question-answer framework seems more intuitive in the context of the approximations.

Proof-theoretical background We work with system $\mathscr{N}\mathscr{T}$ in Table 1, formulated with *signed* formulas of the form $\mathsf{T}A$ or $\mathsf{F}A$, meaning that the agent "holds the information that A is true (respectively, false)" [1, Sec. 2.1]. It results from combining two complete systems for CPL: the refutation system KE and the direct-proof system KI; respectively related to, but more efficient than, Tableaux and truth-tables. Like Natural Deduction,

$\dfrac{\mathsf{T}\,A}{\mathsf{T}\,A \vee B}$	$\dfrac{\mathsf{F}\,A \quad \mathsf{F}\,B}{\mathsf{F}\,A \vee B}$	$\dfrac{\mathsf{F}\,A}{\mathsf{F}\,A \wedge B}$	$\dfrac{\mathsf{T}\,A \quad \mathsf{T}\,B}{\mathsf{T}\,A \wedge B}$	$\dfrac{\mathsf{F}\,A}{\mathsf{T}\,A \to B}$	$\dfrac{\mathsf{T}\,B}{\mathsf{T}\,A \to B}$	$\dfrac{\mathsf{T}\,A \quad \mathsf{F}\,B}{\mathsf{F}\,A \to B}$
$\dfrac{\mathsf{T}\,A}{\mathsf{F}\,\neg A}$	$\dfrac{\mathsf{F}\,A}{\mathsf{T}\,\neg A}$	$\dfrac{\mathsf{T}\,A \vee B \quad \mathsf{F}\,A}{\mathsf{T}\,B}$	$\dfrac{\mathsf{F}\,A \vee B}{\mathsf{F}\,A}$	$\dfrac{\mathsf{F}\,A \wedge B \quad \mathsf{T}\,A}{\mathsf{F}\,B}$	$\dfrac{\mathsf{T}\,A \wedge B}{\mathsf{T}\,A}$	$\dfrac{\mathsf{T}\,A \to B \quad \mathsf{T}\,A}{\mathsf{T}\,B}$
$\dfrac{\mathsf{T}\,A \to B \quad \mathsf{F}\,B}{\mathsf{F}\,A}$	$\dfrac{\mathsf{F}\,A \to B}{\mathsf{T}\,A}$	$\dfrac{\mathsf{F}\,A \to B}{\mathsf{F}\,B}$	$\dfrac{\mathsf{T}\,\neg A}{\mathsf{F}\,A}$	$\dfrac{\mathsf{F}\,\neg A}{\mathsf{T}\,A}$	$\dfrac{\mathsf{T}\,A \vee A}{\mathsf{T}\,A}$	$\dfrac{\mathsf{F}\,A \wedge A}{\mathsf{F}\,A}$
			$\dfrac{}{\mathsf{T}\,A \mid \mathsf{F}\,A}$			

Table 1: $\mathscr{N}\mathscr{T}$ (symmetry of $\vee$ and $\wedge$ is assumed for brevity)

$\mathscr{N}\mathscr{T}$ has introduction and elimination rules, though those that are non-branching and involve only information practically available to the agent and with which she can *operate*. The only branching rule implements the Principle of Bivalence (PB), which allows for the introduction of *hypothetical* information from no premises and thus can be used anywhere in a derivation. An unrestricted application of PB isn't amenable to proof-search, but it can be restricted to applications on the set of *subformulas* of the initial assumptions without affecting completeness. In this regard, the introduction rules can potentially be applied on and on, yielding ever more complex formulas. Yet, they can also be tamed so as to satisfy the subformula property while preserving completeness.

It's in terms of the PB rule that a measure of complexity of derivations is introduced. Namely, a derivation's *depth* is defined as the maximum number of nested applications of PB needed to obtain it. $\mathscr{N}\mathscr{T}$ is advantageous over KE or KI alone, since it reduces the number of PB instances required to obtain a derivation and is closer to human reasoning. This generates a hierarchy of k-depth approximations to CPL, each one tractable whenever the application of PB and the introduction rules are restricted to a suitable subset of formulas as conclusions. Less restrictions on that subset yield deductively more powerful approximations, and tractability crucially depends on appropriate restrictions thereof.

Negative constraints A valuation is a mapping v from any set of formulas Φ to the set of values $\{0, 1, ?\}$, respectively standing for *informational* truth, falsity and indeterminacy. These values are partially ordered by the usual flat relation $\preceq$, defined as $? \preceq x$ and $x \preceq x$ for each $x \in \{0, 1, ?\}$. If v, w are valuations on Φ, then w is a *refinement* of v, , if and only if $v(A) \preceq w(A)$ for all $A \in \Phi$. It is *proper*, if there is a $B \in \Phi$ such that $v(B) \prec w(B)$. We introduce concise notation for refinements consisting of changing the value of a single formula: $v^{A:=x}$ is a refinement of v such that $v^{A:=x}(A) = x$ and $v^{A:=x}(B) = v(B)$ for $A, B \in \Phi$, $B \neq A$, $x \in \{0, 1, ?\ \}$. The classical truth conditions imposed by, e.g. the standard truth-tables, are not suitable for the notion of informational truth. For example, if an agent holds the information that $A \vee B$ is true, she does not necessarily holds the information that A is true or that B is true. Similarly for all cases where the reading of the classical truth-table going from the formula to its components is informationally non-deterministic. This prevents us from giving a direct definition of admissible valuation. Instead, the *negative* constraints expressed in the tables below detect valuations that are *inadmissible* for any agent who 'understands' the informational meaning of the connectives:

A	B	$A \vee B$
0	0	1
1	?	0
?	1	0

A	B	$A \wedge B$
0	?	1
?	0	1
1	1	0

A	B	$A \to B$
1	0	1
0	?	0
?	1	0

A	$\neg A$
1	1
0	0

the main connective of a formula. An agent who understands this meaning can *update* her information state by *uniquely* determining, from her practically available or operational information, the value of formulas that were previously indeterminate. For example, if she holds that $A \wedge B$ is false and B is true, then she can update her state with A being false by complying with the tables for negative constraints, since the refined state with A being true is inadmissible. This 'local' task is computationally and cognitively easy.

Basic game Our semantics is defined in terms of a win-lose perfect information game of two players. An *approximation game* $\mathscr{G}_k(\Gamma, C)$ is given by a set of formulas Γ, a single formula C and parameter $k \in \mathbb{N}$. In the *basic* version of the game, we fix the set of formulas available to the players during the game (the 'game arena') to the set of subformulas of $\Gamma \cup \{C\}$, denoted $\mathsf{sub}(\Gamma \cup \{C\})$. This set constitutes the 'smallest' game board. Now, the goal of the first player called Questioner is to show that C follows from the set of initial assumptions Γ, while the goal of the second player, Responder, is the opposite. For any $i \in \mathbb{N}$, a *state* $S_i = (\Gamma_i, v_i, n_i)$ of a game $\mathscr{G}_k(\Gamma, C)$ is given by a set of formulas Γ_i, a valuation $v_i : \Gamma_i \cup \{C\} \to \{0, 1, ?\}$ and a parameter $n_i \leqslant k$. The valuation v_i represents explicit information currently held by Q, which is updated during the game via admissible refinements. The number n_i is a counter for the number of questions used in the game up to the state S_i.

At the beginning of a game, i.e. at the state S_0, v_0 evaluates all the formulas in Γ to 1, while C and all the formulas in $\mathsf{sub}(\Gamma \cup \{C\}) \setminus \Gamma$ are evaluated to $?$. The value of n_0 is set to 0.

Moves The moves in the approximation game consist of admissible (possibly improper) refinements of a current valuation.

An *answer* consists of determining the value of some currently indeterminate formula B, a task that is only allowed if there is a unique admissible proper refinement determining B. A *question* is an explicit request for a determinate value of a formula A, currently indeterminate. A question itself does not involve a change of the current valuation. An *inadmissibility detection* means that Q found a currently indeterminate formula such that each of the two proper refinements making it determinate is inadmissible. It intuitively corresponds to detecting that the answers given by R, if any, lead to a situation violating the negative constraints.

Formally, a move is a couple lA, such that A is a formula and $l \in \{0, 1, ?, \curlywedge\}$ is a label. Sequences of moves are called *histories*, so $h = l_1 A_1 \dots l_m A_m$. We denote by H the set of all histories and by $h \sqsubseteq h'$ the relation 'h is a subsequence of h''.

Let $S_i = (\Gamma_i, v_i, n_i)$ be the current state of a game and h_i be the current history, that is the history up to S_i. Then a move lA is *legal* in S_i if and only if it has not been played (i.e. $h_j lA \not\sqsubseteq h_i$ with $j < i$) and:

- (**question**) $l = ?$, $n_i < k$, and *at least one* of the proper refinements $v_i^{A:=1}$ and $v_i^{A:=0}$ is admissible, then the game proceeds to $S_{i+1} = (\Gamma_i, v_i, n_i + 1)$; or
- (**answer**) $l \in \{0, 1\}$ and there is a *unique* admissible proper refinement $v_i^{A:=l}$, then the game proceeds to $S_{i+1} = (\Gamma_i \cup \{A\}, v_{i+1}^{A:=l}, n_i)$; or
- (**inadmissibility**) $l = \curlywedge$ and *neither* proper refinement $v_i^{A:=0}$ nor $v_i^{A:=1}$ is admissible, then the game proceeds to $S_{i+1} = S_i$ (and ends).

Player function The roles of players are not symmetric, the possibilities of R are quite restricted, since he cannot ask questions and can answer only if Q explicitly asks. In contrast, Q can ask questions any time and she can also play answers, which can be seen as replies to an 'implicit' question Q asks to herself, thus updating her explicit information. Q can also detect inadmissibility in the sense mentioned above, using the $\curlywedge$-move. Formally, the only histories which are moves for R are those of the form $h?A$ for some $h \in H$. All the other histories, including the empty one, are moves for Q.

End of the game Winning conditions for Q are simple, either the valuation of the conclusion C is set to 1 in some move (either by Q or by R) or she detects an inadmissibility. R wins if Q cannot move any more, which includes the case when she has spent all her k questions. In contrast, setting the conclusion false does not suffice, for Q might still spot inadmissibility on a formula in the remaining part of the game.

Formally, h_i is a terminal history and S_i is a terminal state of the game $\mathscr{G}_k(\Gamma, C)$ iff:

- (**conclusion true**) $h_i = h_{i-1}1C$ and consequently $v_i(C) = 1$;
- (**inadmissibility detected**) $h_i = h_{i-1} \curlywedge A$, and thus neither proper refinement with $v_i(A) = 0$ nor $v_i(A) = 1$ of v_i is admissible;
- (**no moves**) $h_i \neq h_{i-1}?A, n_i = k$, and there is no $B \in \Gamma_i$ for which there is a *unique* proper admissible refinement $v_i^{B:=l}$, $l \in \{0, 1\}$.

There are no special *procedural* rules in the basic version of the game. Q starts the game and then plays answers and questions in an arbitrary order as long as she can, i.e. until the end of the game is reached. R moves only when asked a question. No player can repeat her/his moves according to the definition of a legal move.

Correspondence theorem We show that here is a k-depth proof of $\mathsf{T}C$ from $\mathsf{T}\Gamma$ in $\mathscr{N}\mathscr{T}$ over the sub-bounded ('analytic') *search space* [1, 2.1] if and only if there is a winning strategy for Q in the basic game $\mathscr{G}_k(\Gamma, C)$.

Adequacy and intuitiveness Some natural *liberalizations* of the basic version of the game are, for example: (i) setting a wider 'game arena', in particular, allowing *questions* from a superset of $\mathsf{sub}(\Gamma \cup \{C\})$; (ii) allowing for controlled move repetition. By contrast, some natural *restrictions* are, for instance: (i) Q asks *only* when she cannot answer herself (corresponding to pushing PB as down as possible); (ii) questions restricted to atomic formulas. In any case, a remarkable intuition is that questions are a resource worth keeping! Actually, it is exactly when Q exploits the information she holds as much as possible that the answers of R correspond only to the introduction of information that was not even implicitly contained in the information held by Q.

Thoughtful questions correspond to resource savings, while hasty questions correspond to a wasteful play. Indeed, question selection is not a trivial task, whose difficulty increases proportionally with the number of questions needed and the freedom on the subset from which these are selected. The levels of the hierarchy of approximations, under proof-theoretic restrictions, can intuitively be associated with increasingly better questioners, in terms of their 'ingenuity' when selecting questions, under suitable playing freedom. More liberalized settings are naturally associated with more competent questioners, and thus with more efficient playing in that the number of questions needed can dramatically decrease.

Strategies by the Questioner arise naturally when balancing her question 'budget' with playing freedom and competence thereof. These strategies intuitively correspond to different *procedures* when implementing the background proof-theory.

Finally, we envisage *extensions* of our semantics to non-classical depth-bounded approximations, such as FDE [2] and IPL, by modifying our game in the spirit of dialogics.

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Conditionals as quotients in Boolean algebras

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In this contribution we introduce and study a logico-algebraic notion of conditional operator. A conditional statement is a hypothetical proposition of the form

"If [antecedent] is the case, then [consequent] is the case",

where the antecedent is assumed to be true. Such a notion can be formalized by expanding the language of classical logic by a binary operator a/b that reads as "a given b".

A most well-known approach in this direction comes from a philosophical perspective developed first by Stalnaker [9, 10], and further analyzed by Lewis [5], that in order to axiomatize the operator / ground their investigation on particular Kripke-like structures. In particular, Lewis defines a hierarchy of logics for conditionals, which have been shown to be algebraizable in [7] with respect to varieties of Boolean algebras with operators, named *Lewis variably strict conditional algebras* or *V-algebras*.

The novel approach we propose here is grounded in the algebraic setting of Boolean algebras, where we show that there is a natural way of formalizing conditional statements.

The algebraic intuition

Given a Boolean algebra $\mathbf{B}$ and an element b in B, one can define a new Boolean algebra, say $\mathbf{B}/b$, intuitively obtained by assuming that b is true. More in details, one considers the congruence collapsing b and the truth constant 1, and then $\mathbf{B}/b$ is the corresponding quotient. Then the idea is to define a conditional operator / such that a/b represents the element a as seen in the quotient $\mathbf{B}/b$, mapped back to $\mathbf{B}$. The particular structural properties of Boolean algebras allow us to do so in a natural way.

Note that if $b \neq 0$ the quotient $\mathbf{B}/b$ is actually a *retract* of $\mathbf{B}$, which means that if we call π_b the natural epimorphism $\pi_b : \mathbf{B} \to \mathbf{B}/b$, there is an injective homomorphism $\iota_b : \mathbf{B}/b \to \mathbf{B}$ such that $\pi_b \circ \iota_b$ is the identity map.

The idea is then to consider

$$a/b := \iota_b \circ \pi_b(a). \tag{0.1}$$

We observe that the map ι_b is not uniquely determined, meaning that there can be different injective homomorphisms ι, ι' such that $\pi_b \circ \iota = \pi_b \circ \iota'$ is the identity; distinct choices yield distinct values for a/b.

Now, in order to be able to define an operator / over the algebra $\mathbf{B}$, one needs to consider all the different quotients, determined by all choices of elements $b \in B$. Then, if $0 \neq b \leqslant c$, by general algebraic arguments one gets a natural way of looking at nested conditionals; indeed it holds that $(\mathbf{B}/c)/\pi_c(b) = \mathbf{B}/b$, which means that $\mathbf{B}/b$ is a quotient of $\mathbf{B}/c$, and actually also its retract. It is then natural to ask that the choices for ι_b and ι_c be *compatible*, in the sense that there is a way of choosing the embedding $\iota_{\pi_c(b)}$ so that

$$\iota_b = \iota_c \circ \iota_{\pi_c(b)}, \tag{0.2}$$

which yields in particular that $a/b = (a/b)/c$ whenever $b \leqslant c$.

The case where $b = 0$ needs to be considered separately, since the associated quotient is the trivial algebra that cannot be embedded back into $\mathbf{B}$. Since intuitively we are considering the quotients by an element b to mean that "b is true", the *ex falso quodlibet* suggests that we map all elements to 1, i.e:

$$a/0 := 1. \tag{0.3}$$

The idea is then to use Stone duality to translate the above conditions to the dual setting; in other words, we generate the intended models as algebras of sets.

The standard models via Stone duality

For simplicity, let us describe our setup restricting to finite algebras. By the finite version of Stone duality, we now see the algebra $\mathbf{B}$ as an algebra of sets, say that $\mathbf{B} = \mathscr{S}(X)$ for a set X. Then the above reasoning translates to the following.

Given $Y \subseteq X$, the natural epimorphism $\pi_Y : \mathscr{S}(X) \to \mathscr{S}(Y)$ dualizes to the identity map $\mathrm{id}_Y : Y \to X$, and the embedding $\iota_Y : \mathscr{S}(Y) \to \mathscr{S}(X)$ dualizes to a surjective map $f_Y : X \to Y$, such that $f_Y \circ \mathrm{id}_Y = \mathrm{id}_Y$; in other words, we are asking that f_Y restricted to Y is the identity. Moreover, consider $Y \subseteq Z \subseteq X$. Then the compatibility condition (0.2) becomes on the dual $f_Y = f_Y^Z \circ f_Z$, where f_Y^Z is the dual of the map $\iota_{\pi_Z(Y)}$. The standard models are those that originate by the above postulates; let us be more precise.

Definition 1. Given a set X, we say that a class of surjective functions $\mathscr{F} = \{f_Y^Z : Z \to Y : \varnothing \neq Y \subseteq Z \subseteq X\}$ with $f_Y^Z : Z \to Y$ is *compatible with* X if:

1. f_Y^X restricted to Y is the identity on Y;
2. $f_Y^X = f_Y^Z \circ f_Z^X$.

We now define the class of standard models as algebras of sets.

Definition 2. A *standard model* is an algebra with operations $\{\wedge, \vee, \neg, /, 0, 1\}$ that is a Boolean algebra of sets $\mathscr{S}(X)$ for some set X with $/$ defined as follows from a class of functions $\mathscr{F}$ compatible with X:

$$Y/Z := (f_Z^X)^{-1}(Y \cap Z)$$

for any $Y, Z \subseteq X$ and $Z \neq \varnothing$, and for any $Y \subseteq X$ we set $Y/\varnothing := X$.

The variety of Quonditional Algebras

Let us call the variety of algebras generated by the standard models QA, and the algebras therein Quonditional Algebras.

In this work, we prove that QA is a subvariety of the variety of Lewis variably strict conditional algebras VA. In particular, with respect to VA one needs to add the algebraic identities characterizing the models of Stalnaker's logic of conditionals [5, 9, 7]:

$$x \wedge y \ \leqslant\ y/x \ \leqslant\ x \to y; \quad y/x \ \vee\ \neg y/x = 1,$$

plus the algebraic version of a condition called *uniformity* in [5], and the axiom arising from Condition 0.2:

$$x/y = (x/y)/(y \vee z).$$

Interestingly, this last axiom, which here arises from a purely algebraic compatibility condition, has been discussed in the literature through the name of *flattening* in [1]; there it is used to axiomatize a logic of conditionals introduced in [3], which introduced a particularly simple special case of ordering semantics for conditionals based on functions from the natural numbers to the set of possible worlds.

We then provide an algebraic study of this variety, which in particular turns out to be a *discriminator variety*. In a variety of algebras with a Boolean reduct like in this case, this means that the variety is generated by algebras with a unary term t such that

$$t(0) = 0 \quad \text{and} \quad t(x) = 1, \text{ if } x \neq 0;$$

specifically, we show that all standard models have as discriminator term $t(x) = \neg(0/x)$.

The fact that QA is a discriminator variety, in particular entails that the classes of subdirectly irreducible, directly indecomposable, and simple algebras in QA coincide, and in this case they are exactly the class of (isomorphic copies of) standard models.

Finally, we study the duality theory for QA, in terms of Stone spaces with a ternary relation. With this respect, we observe that the binary operator $/$ is not simply an application of Jonnson-Tarski duality for Boolean algebras with (modal) operators [4]; indeed, the models are not Boolean algebras with an operator in the usual sense, since $/$ is not additive on both arguments (more precisely, it only distributes over meets on the consequent) and it cannot be recovered from a unary modal operator. In particular, we base this analysis on the duality for VA presented in [8].

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Projective Unification and Structural Completeness in Extensions and Fragments of Bi-Intuitionistic Logic

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Bi-Intuitionistic Logic

Definition 1. Let $\mathscr{L} := (\wedge, \vee, \to, \leftarrow, \neg, \sim, \bot)$ be a formal algebraic language of signature $(2, 2, 2, 2, 1, 1, 0)$, called *bi-intuitionistic language*. Given a sublanguage $\mathscr{L}_1 \subseteq \mathscr{L}$, we denote its set of formulas built up from a denumerable set of variables $\{p, q, \dots\}$ by $Fm_{\mathscr{L}_1}$. If $\varphi \in Fm_{\mathscr{L}_1}$, then $var(\varphi)$ denotes the set of variables occurring in φ. A *logic* in $\mathscr{L}_1$ is a finitary consequence relation $\vdash$ on $Fm_{\mathscr{L}_1}$ that is also substitution invariant. We write $\Gamma \vdash \varphi$ instead of $(\Gamma, \varphi) \in \vdash$. If $\varnothing \vdash \varphi$ then φ is called a *theorem* of $\vdash$, and we will use the shorthand notation $\vdash \varphi$.

Let $\mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \mathscr{L}$ be sublanguages. If $\vdash_1$ is a logic in $\mathscr{L}_1$, then the $\mathscr{L}_0$*-fragment* of $\vdash_1$ is the restriction of $\vdash_1$ to $Fm_{\mathscr{L}_0}$. If $\vdash_0$ is a logic in $\mathscr{L}_0$, then $\vdash_1$ is an *extension* of $\vdash_0$ if $\vdash_0 \subseteq \vdash_1$. If moreover $\mathscr{L}_0 = \mathscr{L}_1$ and there exists $\Sigma \subseteq Fm_{\mathscr{L}_0}$ such that

$$\Gamma \vdash_1 \varphi \iff \Gamma \cup \Sigma \vdash_0 \varphi$$

for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathscr{L}_0}$, we call $\vdash_1$ an *axiomatic extension* of $\vdash_0$. In the case that $\Sigma = \{\psi\}$, we will sometimes write $\vdash_1 = \vdash_0 + \psi$.

Bi-intuitionistic logic bi-IPC is the conservative extension of intuitionistic logic IPC obtained by enlarging the intuitionistic language with the connectives $\leftarrow$ and $\sim$, called *co-implication* and *co-negation*, and demanding that they behave dually to $\to$ and $\neg$, respectively. In this way, bi-IPC achieves a symmetry, which IPC lacks, between the connectives $\wedge, \to, \neg, \bot$ and $\vee, \leftarrow, \sim, \top$. The Kripke semantics of bi-IPC [10] provides a transparent interpretation of co-implication: given a Kripke model $\mathfrak{M}$, a point x in $\mathfrak{M}$, and formulas φ, ψ, then

$$\mathfrak{M}, x \models \varphi \leftarrow \psi \iff \exists y \leqslant x\ (\mathfrak{M}, y \models \varphi \text{ and } \mathfrak{M}, y \not\models \psi).$$

Using this equivalence, the intended behavior of co-negation also becomes clear, because the formula $\sim p \leftrightarrow (\top \leftarrow p)$ is a theorem of bi-IPC.

Equipped with these new connectives, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. In fact, Gödel's interpretation of IPC into the modal logic S4 can be extended to an interpretation of bi-IPC into the temporal modal logic tense-S4 [12].

The greater symmetry of bi-IPC with respect to IPC is reflected by the fact that bi-IPC is algebraized in the sense of [4] by the variety bi-HA of *bi-Heyting algebras* [9], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, we can (and will) identify bi-IPC with the logic induced by the class of matrices $\{\langle \mathbf{A}, \{1\}\rangle \colon \mathbf{A} \in \text{bi-HA}\}$, and sometimes denote it by $\vdash_{\text{bi-IPC}}$. Notably [6], for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathscr{L}}$ we have

$$\Gamma \vdash_{\mathfrak{bi\text{-}IPC}} \varphi \iff \text{if } \mathfrak{M} \text{ is a Kripke model, then } \mathfrak{M} \models \Gamma \text{ implies } \mathfrak{M} \models \varphi.$$

Projective Unification

Let $\mathscr{L}_0$ be a sublanguage of the bi-intuitionistic language $\mathscr{L}$ and $\vdash$ a logic in $\mathscr{L}_0$. A formula $\varphi \in Fm_{\mathscr{L}_0}$ is said to be *unifiable* in $\vdash$ if we have $\vdash \sigma(\varphi)$, for some substitution σ. In this case, σ is called a $\vdash$-*unifier* of φ, or simply a *unifier* of φ, when the logic $\vdash$ is clear from the context. If moreover the language $\mathscr{L}_0$ contains the connectives $\mathscr{L}nd$ and $\to$, and $\varphi \vdash p \leftrightarrow \sigma(p)$ holds for every $p \in var(\varphi)$, then σ is called a *projective unifier* of φ.

If σ and τ are two unifiers of φ, we say that σ is *at least as general as* τ, denoted by $\sigma \leqslant \tau$, if there exists a substitution μ such that $\vdash \sigma(p) \leftrightarrow \mu \circ \tau(p)$, for every $p \in var(\varphi)$. A set E of unifiers of φ is said to be a *basis* if: for every unifier τ of φ, there exists $\sigma \in E$ such that $\sigma \leqslant \tau$; and for all $\sigma, \sigma' \in E$, if $\sigma \leqslant \sigma'$ then $\sigma = \sigma'$. In particular, if $E = \{\sigma\}$ is a one-element basis, then σ is called a *most general unifier* of φ. It is easy to see that a projective unifier of φ is always a most general unifier of φ. We call φ *unitary* if it admits a most general unifier and *projective* if it admits a projective unifier. Accordingly, the logic $\vdash$ is said to be *unitary* (resp. *projective*) if every unifiable formula is unitary (resp. projective).

In the first part of this talk, I will present an unpublished joint work with Damiano Fornasiere and Quentin Gougeon, where we characterized the projective *bi-intermediate logics* (i.e., consistent axiomatic extensions of bi-IPC): they are exactly those which have a theorem of the form[1] $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$, for some $n \in \omega$. Compare this to [13], where it is shown that the projective *intermediate logics* (i.e., consistent axiomatic extensions of IPC) are exactly those which extend the *Gödel-Dummett logic* GD $:=$ IPC $+ (p \to q) \vee (q \to p)$. And although being an extension of the *bi-intuitionistic Gödel-Dummett logic* bi-GD $:=$ bi-IPC $+ (p \to q) \vee (q \to p)$ is a sufficient condition for a bi-intermediate logic to

[1] For $n \in \omega$ and $\varphi \in Fm_{\mathscr{L}}$, we define $(\neg\sim)^n \varphi$ recursively by $(\neg\sim)^0 \varphi := \varphi$ and $(\neg\sim)^{n+1} \varphi := \neg(\sim(\neg\sim)^n \varphi)$.

be projective (because $\neg\sim p \to (\neg\sim)^2 p$ is a theorem of bi-GD, see [2]), it is not necessary. For example, since $(\neg\sim)^2 p \to (\neg\sim)^3 p$ is a theorem of bi-IPC $+ \neg\big((q \leftarrow p) \wedge (p \leftarrow q)\big)$, our characterization ensures that this bi-intermediate logic is projective, but it is not an extension of bi-GD [2].

Semantically, bi-intermediate logics with a theorem of the form $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$ can be characterized by the property of having a natural bound for the zigzag depth of the Kripke frames which validate them, a notion that we proceed to explain. If u and v are points in a Kripke frame $\mathfrak{F}$, we say that v *can be reached from* u *after* n*-many zigzags* if there are $x_1, y_1, \ldots, x_n \in \mathfrak{F}$ such that $u \leqslant x_1 \geqslant y_1 \leqslant x_2 \geqslant y_2 \leqslant \ldots \leqslant x_n \geqslant v$. We then define, for every $U \subseteq \mathfrak{F}$, the set $(\downarrow\uparrow)^n U$ of points of $\mathfrak{F}$ that can be reached from a point in U after n-many zigzags. Notably, if $\mathfrak{M} = (\mathfrak{F}, V)$ is a Kripke model on $\mathfrak{F}$, then $V\big((\neg\sim)^n \varphi\big) = (\downarrow\uparrow)^n V(\varphi)$ for all $\varphi \in Fm_{\mathscr{L}}$. Using this equality, one can easily show that

$$\mathfrak{F} \models (\neg\sim)^n p \to (\neg\sim)^{n+1} p \iff \uparrow(\downarrow\uparrow)^n U = (\downarrow\uparrow)^{n+1} U \text{ for every upset } U \text{ of } \mathfrak{F},$$

and when these conditions are satisfied, we say that $\mathfrak{F}$ has n*-bounded zigzag depth*.

That a bi-intermediate logic $\vdash$ with a theorem of the form $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$ must be projective already follows from the literature: in [11], it is shown that such logics are exactly those with a *discriminator term*, whereas in [5], it is established that for an algebraizable logic (in particular, for all bi-intermediate logics), having a discriminator term is a sufficient condition for projectivity.

In order to prove the converse, i.e., that any projective bi-intermediate logic $\vdash$ must contain a theorem of the form $(\neg\sim)^n p \to (\neg\sim)^{n+1} p$, we introduce for each $n \in \omega$ the formula

$$\theta_n := (\neg\sim)^{n+2} p \to (\neg\sim)^{2n+4} \neg(\neg\sim)^n \neg p,$$

and prove that $\vdash \theta_n$ forces $\vdash (\neg\sim)^{2n+3} p \to (\neg\sim)^{2n+4} p$. We then assume that $\vdash$ is projective, so the fact that the formula $\varphi := p \to \neg\sim p$ is unifiable in $\vdash$ (simply take a substitution that sends p to $\top$) entails that it must have a projective unifier σ. It follows that $\varphi \vdash p \leftrightarrow \sigma(p)$. By using the Deduction Theorem for bi-intermediate logics [6], which states that

$$\Gamma, \varphi \vdash \psi \iff \exists n \in \omega \,\big(\Gamma \vdash (\neg\sim)^n \varphi \to \psi\big),$$

for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathscr{L}}$, we infer that $\vdash (\neg\sim)^n \varphi \to \big(p \leftrightarrow \sigma(p)\big)$, for some $n \in \omega$. Then, with a view to contradiction, we assume that $\nvdash (\neg\sim)^{2n+3} p \to (\neg\sim)^{2n+4} p$, hence $\nvdash \theta_n$ by above. After some semantical combinatorics, the aforementioned consequence of the Deduction Theorem, together with $\nvdash \theta_n$, is enough to arrive at the desired contradiction.

We also showed that bi-IPC is not unitary, by proving that while the formula $p \to \neg\sim p$ is unifiable in bi-IPC, it does not admit a most general unifier.

Structural Completeness

Let $\mathscr{L}_0$ be a sublanguage of the bi-intuitionistic language $\mathscr{L}$ and $\vdash$ a logic in $\mathscr{L}_0$. A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathscr{L}_0}$ is finite. A rule $\Gamma \rhd \varphi$ is said to be *valid* in $\vdash$ if $\Gamma \vdash \varphi$, and *admissible* in $\vdash$ if for every substitution σ we have that $\vdash \sigma[\Gamma]$ implies $\vdash \sigma(\varphi)$ (that is, if a substitution σ is a $\vdash$-unifier of all the formulas in Γ, then it must also be a $\vdash$-unifier of φ). We denote by $\vdash + \Gamma \rhd \varphi$ the least (wrt. inclusion) logic in $\mathscr{L}_0$ containing $\vdash \cup (\Gamma, \varphi)$. The logic $\vdash$ is said to be *structurally complete* if every admissible rule is also valid (the converse holds in general, by substitution invariance). We denote the least (wrt. inclusion) structurally complete logic in $\mathscr{L}_0$ containing $\vdash$ by $Sc(\vdash)$, and call it the *structural completeness* of $\vdash$.

Let $\mathsf{bi\text{-}IPC}^-$ be the $(\wedge, \vee, \neg, \sim)$-fragment of $\mathsf{bi\text{-}IPC}$. A standard and straightforward argument shows that this is the logic induced by the class of matrices $\{\langle \mathbf{A}, \{1\}\rangle \colon \mathbf{A} \in \mathsf{bi\text{-}PDL}\}$, where $\mathsf{bi\text{-}PDL}$ denotes the variety of *double pseudocomplemented distributive lattices*. Notably, this class of algebras enjoys a restricted version of the celebrated *Priestley duality* that associates to each $\mathbf{A} \in \mathsf{bi\text{-}PDL}$ its *dual bi-p-space* $\mathbf{A}_*$ (see, e.g., [4]). And conversely, to each bi-p-space $\mathscr{X}$ we associate its *double pseudocomplemented dual* $\mathscr{X}^*$. Using this duality, one can show that the class $Mod^*(\mathsf{bi\text{-}IPC}^-)$ of reduced matrix models of $\mathsf{bi\text{-}IPC}^-$ satisfies the equality

$$Mod^*(\mathsf{bi\text{-}IPC}^-) = \{\langle \mathbf{A}, \{1\}\rangle \colon \mathbf{A} \in \mathsf{bi\text{-}PDL} \text{ and } dp(\mathbf{A}_*) \leqslant 2\},$$

where $dp(\mathbf{A}_*)$ denotes the *depth* of the underlying poset of $\mathbf{A}_*$.

In the second part of this talk, I will present an unpublished joint work with Tommaso Moraschini, where we proved that, except for the classical propositional calculus CPC, no consistent and locally finite axiomatic extension of $\mathsf{bi\text{-}IPC}^-$ is structurally complete[2]. This result is in sharp contrast with [7], where it is shown that every axiomatic extension of the $(\wedge, \vee, \neg)$-fragment of IPC must be structurally complete.

Let $\vdash$ be a fixed but arbitrary locally finite axiomatic extension of $\mathsf{bi\text{-}IPC}^-$. If $\mathscr{X}$ is a bi-p-space, we write $\mathscr{X} \in Mod(\vdash)$ when $\langle \mathscr{X}^*, \{1\}\rangle \in Mod(\vdash)$ holds true. Using the equality in the previous display, we prove that the class $FgMod^*(\vdash)_{RSI}$ of finitely generated relatively subdirectly irreducible reduced matrix models of $\vdash$ can be identified with

$$\{\mathbf{A} \in \mathsf{bi\text{-}PDL} \colon \mathbf{A}_* \in Mod(\vdash) \text{ and } \mathbf{A}_* \text{ is a finite connected poset of depth} \leqslant 2\}.$$

We then interpret [8, Thm. 2.12] within our setting. This result establishes equivalent conditions for a rule to be admissible in a logic over an arbitrary algebraic language. By making use of the properties of bi-p-spaces and their morphisms, and the fact that $\vdash$ was assumed to be locally finite, we derive from the aforementioned interpretation many

[2] In [1], it is proved that apart from CPC, every bi-intermediate logic is not structurally complete. However, structural completeness results are very sensitive to changes in signature. Moreover, our methods diverge significantly, because unlike all the bi-intermediate logics, the axiomatic extensions of $\mathsf{bi\text{-}IPC}^-$ are not algebraizable.

equivalent conditions for a rule to be admissible in $\vdash$, culminating in the following[3]: the rule $\Gamma \rhd \varphi$ is admissible in $\vdash$ iff for every finite connected poset $\mathscr{X}$ of depth $\leqslant 2$ such that $\mathscr{X} \in Mod(\vdash)$, there exists $\mathscr{Y}$, a finite poset of depth $\leqslant 2$ satisfying $\mathscr{Y} \in Mod(\vdash + \Gamma \rhd \varphi)$, and such that $\mathscr{X}^*$ is a homomorphic image of $\mathscr{Y}^*$. Finally, we use the previous equivalence to show that $Sc(\vdash)$, the structural completeness of $\vdash$, coincides with $Log(\mathbf{K}_\vdash)$, the logic induced by the class of matrices

$$\mathbf{K}_\vdash := \{\left\langle(\mathscr{X} \uplus \bullet)^*, \{1\}\right\rangle : \mathscr{X} \in Mod(\vdash) \text{ and } \mathscr{X} \text{ is a finite connected poset of depth} \leqslant 2\},$$

where $\mathscr{X} \uplus \bullet$ denotes the disjoint union of a poset $\mathscr{X}$ with a singleton poset. Then, a semantical argument ensures that if $FgMod^*(\vdash)_{RSI}$ contains a matrix $\langle \mathbf{A}, \{1\}\rangle$ such that the dual bi-p-space $\mathbf{A}_*$ is not a singleton (which is the case for every consistent axiomatic extension of bi-IPC$^-$ distinct from CPC), we have that $\vdash \subsetneqq Log(\mathbf{K}_\vdash) = Sc(\vdash)$, i.e, that $\vdash$ is not structurally complete. We are currently working on proving the analogous result for arbitrary (i.e., not necessarily locally finite) axiomatic extensions of bi-IPC$^-$.

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Recent Advances in Fundamental Logic

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Holliday [3] introduced a non-classical logic called *fundamental logic*, which captures exactly those properties of the connectives $\wedge, \vee$ and $\neg$ that hold in virtue of their introduction and elimination rules in Fitch's natural deduction system for propositional logic. Fundamental logic is a sublogic of both (the $\rightarrow$-free fragment of) intuitionistic logic and orthologic. The former can be obtained from fundamental logic by adding the *Reiteration* rule to Holliday's Fitch system for fundamental logic, while the second can be obtained by adding the *Double Negation Elimination* rule.

From the algebraic perspective, fundamental logic is the logical counterpart to the variety of fundamental lattices:

Definition 1. A *fundamental lattice* is a tuple $(L, \leqslant, \wedge, \vee, \neg, 0, 1)$ such that $(L, \leqslant, \wedge, \vee, 0, 1)$ is a bounded lattice and $\neg : L \to L$ is an antitone map satisfying the following properties:

- $\neg 1 = 0$;
- $a \wedge \neg a = 0$;
- $a \leqslant \neg\neg a$.

Since fundamental logic is weaker than both intuitionistic logic and orthologic, fundamental lattices generalize both pseudocomplemented distributive lattices and ortholattices.

In this talk based on joint projects with Wes Holliday and Juan P. Aguilera respectively, I will present some recent results which shed some new light on the relationship between fundamental logic, intuitionistic logic and orthologic.

First, I will discuss two translations of fundamental logic into modal orthologic and modal intuitionistic logic. The first translation is based on the celebrated Gödel-McKinsey-Tarski translation [4] of intuitionistic logic into $\mathsf{S4}$, the modal logic of reflexive and transitive Kripke frames. The restriction of this translation to the $\rightarrow$-free fragment of IPC is a map τ inductively defined as follows:

$$\tau(p) = \Box p;$$

$$\tau(\neg\varphi) = \Box\neg\tau(\varphi);$$

$$\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi);$$

$$\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi).$$

As it turns out, this translation also yields an embedding of fundamental logic into $\mathsf{OS4}$, the natural counterpart of $\mathsf{S4}$ in orthomodal logic.

Theorem 1 (Holliday and Massas 2025)**.** *The Gödel-McKinsey-Tarski translation τ is a full and faithful translation of fundamental logic into* $\mathsf{OS4}$*.*

A similar result can be obtained by "swapping" the roles of intuitionistic logic and ortho-logic. Goldblatt [2] defined the following translation σ from the language of orthologic into the language of modal logic:

$$\sigma(p) = \Box\Diamond p;$$

$$\sigma(\neg\varphi) = \Box\neg\sigma(\varphi);$$

$$\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi);$$

$$\sigma(\varphi \vee \psi) = \Box\Diamond(\sigma(\varphi) \vee \sigma(\psi)).$$

Goldblatt shows that σ is a full and faithful translation of orthologic into the modal logic KTB of reflexive and symmetric Kripke frames. In order to generalize this result, we define the logic FSTB, a natural counterpart of KTB in the setting of Fischer Servi intuitionistic modal logics [1].

Definition 2. The intuitionistic modal logic FSTB extends the Fischer Servi logic FS with the following axioms:

$$\Box\varphi \vdash \varphi,\ \varphi \vdash \Diamond\varphi;$$

$$\Diamond\Box\varphi \vdash \varphi,\ \varphi \vdash \Diamond\Box\varphi.$$

Theorem 2 (Holliday and Massas 2025)**.** *The Goldblatt translation σ is a full and faithful translation of fundamental logic into* FSTB*.*

These results establish that fundamental logic is, arguably, both "intuitionistic logic from the viewpoint of orthologic", and "orthologic from the viewpoint of intuitionistic logic".

Lastly, I will discuss the relationship between fundamental logic and orthointuitionistic logic, i.e., the strongest logic contained in both the $\to$-free fragment of intuitionistic logic and orthologic. Although fundamental logic is strictly weaker than orthointuitionistic logic, the latter turns out to have a reasonably simple axiomatization.

Definition 3. Let Ex be the smallest consequence relation extending fundamental logic and closed under the following axioms:

$$\neg\neg p \wedge \neg\neg q \vdash \neg\neg(p \wedge q); \tag{Nu}$$

$$\neg\neg p \wedge q \wedge (r \vee s) \vdash p \vee (q \wedge r) \vee (q \wedge s); \tag{Vi}$$

$$\neg(p \wedge ((q \wedge r) \vee (q \wedge s))) \wedge p \vdash (q \wedge (r \vee s)) \vee \neg(q \wedge (r \vee s)). \tag{Cl}$$

Theorem 3 (Aguilera and Massas 2025)**.** *The logic* Ex *is the strongest extension of fundamental logic that is weaker than both orthologic and intuitionistic logic.*

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Difference–restriction algebras with operators

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Abstract

We exhibit an adjunction between a category of abstract algebras of partial functions that we call difference–restriction algebras and a category of Hausdorff étale spaces. Difference–restriction algebras are those algebras isomorphic to a collection of partial functions closed under relative complement and domain restriction; the morphisms are the homomorphisms. Our adjunction generalises the adjunction between the generalised Boolean algebras and the category of Hausdorff spaces. We define the finitary compatible completion of a difference–restriction algebra, and show that the monad induced by our adjunction yields the finitary compatible completion of any difference–restriction algebra. The adjunction restricts to a duality between the finitarily complete difference–restriction algebras and the locally compact zero-dimensional Hausdorff étale spaces, generalising the duality between generalised Boolean algebras and locally compact zero-dimensional Hausdorff spaces. We then extend these adjunction, duality, and completion results to difference–restriction algebras equipped with arbitrary additional compatibility preserving operators.

Introduction

The study of algebras of partial functions is an active area of research that investigates collections of partial functions and their interrelationships from an algebraic perspective. The partial functions are treated as abstract elements that may be combined algebraically using various natural operations such as composition, domain restriction, 'override', or 'update'. In pure mathematics, algebras of partial functions arise naturally as structures such as inverse semigroups, pseudogroups, and skew lattices. In theoretical computer science, they appear in the theories of finite state transducers, computable functions, deterministic propositional dynamic logics, and separation logic. Many different selections of operations have been considered, each leading to a different category of abstract algebras (see [9, $\mathscr{S}$ 3.2] for a guide). Recently, dualities for some of these categories have started to appear [7, 6, 8, 10, 1], opening the way for these algebras to be studied via their duals.

In [2] and [3], we initiated a project to develop a general and modular framework for producing and understanding dualities for such categories. For this we are inspired strongly by Jónsson and Tarski's theory of Boolean algebras with operators [5] and the duality between them and descriptive general frames. Our central thesis is that in our case the appropriate base class—the analogue of Boolean algebras—must be more than just a class of *ordered* structures but must record additional *compatibility* data. This reflects the fact that the union of two partial functions is not always a function.

In [2] and [3], we investigated algebras of partial functions for a signature we believe provides the necessary order and compatibility structure. The signature has two operations: set-theoretic *relative complement* and a *domain restriction* operation $\triangleright$ given by: $f \triangleright g := \{(x, y) \in X \times Y \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}$. In [2], we gave and proved a finite equational axiomatisation for the class of isomorphs of such algebras of partial functions [2, Theorem 5.7]. We will refer to the algebras in this class as *difference–restriction algebras*. In [3], we gave a 'discrete' duality between the *atomic* difference–restriction algebras and a category of set quotients.

The main results of the present work are the elaboration of an adjunction between the category of difference–restriction algebras and the category of Hausdorff étale spaces (Thm. 1) and the extension of that theorem to algebras with additional operators (Thm. 3). We also show the monad induced by the adjunction gives a form of finitary completion of algebras (Thm. 2/Cor. 1(i)) and the adjunction restricts to a duality between the finitarily complete algebras and the locally complete zero-dimensional Hausdorff étale spaces (Thm. 1/Cor. 1(ii)).

Difference–restriction algebras and adjunction

An **algebra of partial functions** of the signature $\{-, \triangleright\}$ is a $\{-, \triangleright\}$-algebra whose elements are partial functions from some (common) set X to some (common) set Y, and the interpretation of $-$ is *relative complement* and the interpretation of $\triangleright$ is *domain restriction*.

A **difference–restriction algebra** is an algebra $\mathfrak{A}$ of the signature $\{-, \triangleright\}$ that is isomorphic to an algebra of partial functions. We denote by **DRA** the category whose objects are difference–restriction algebras and whose morphisms are homomorphisms of $\{-, \triangleright\}$-algebras. The operation $a \cdot b := (a - (a - b))$ gives difference–restriction algebras a semilattice structure.

An **étale space** is a surjective local homeomorphism $\pi\colon X \twoheadrightarrow X_0$ (i.e., each $x \in X$ has an open neighbourhood U on which π restricts to a homomorphism, and $\pi(U)$ is open), and π is **Hausdorff** if X is Hausdorff. A partial function $\varphi\colon X \rightharpoonup Y$ is **continuous** if when $V \subseteq Y$ is open in Y then $\varphi^{-1}(V)$ is open in X, and φ is **proper** if whenever $V \subseteq Y$ is compact then $\varphi^{-1}(V)$ is compact.

Definition 1. We denote by **HausEt** the category whose objects are Hausdorff étale spaces $\pi\colon X \twoheadrightarrow X_0$, and where a morphism from $\pi\colon X \twoheadrightarrow X_0$ to $\rho\colon Y \twoheadrightarrow Y_0$ is a continuous and proper partial function $\varphi\colon X \rightharpoonup Y$ satisfying the following conditions:

(Q.1) φ **preserves equivalence**: if both $\varphi(x)$ and $\varphi(x')$ are defined, then $\pi(x) = \pi(x') \implies \rho(\varphi(x)) = \rho(\varphi(x'))$; thus there is an induced $\widetilde{\varphi}\colon X_0 \rightharpoonup Y_0$,

(Q.2) φ is **fibrewise injective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the restriction and co-restriction of φ induces an injective partial map $\varphi_{(x_0,y_0)}\colon \pi^{-1}(x_0) \rightharpoonup \rho^{-1}(y_0)$,

(Q.3) φ is **fibrewise surjective**: for every $(x_0, y_0) \in \widetilde{\varphi}$, the induced partial map $\varphi_{(x_0,y_0)}$ is surjective (that is, the image of $\varphi_{(x_0,y_0)}$ is the whole of $\rho^{-1}(y_0)$).

Theorem 1. *There exist adjoint functors $F\colon$ **DRA** $\to$ **HausEt**$^{\mathrm{op}}$ and $G\colon$ **HausEt**$^{\mathrm{op}}$ $\to$ **DRA**.*

Roughly, F is 'maximal filters' and G is 'partial sections with compact image'.

Duality and completion

Two elements of a difference–restriction algebra are **compatible** if $a_1 \triangleright a_2 = a_2 \triangleright a_1$ (corresponding to partial functions agreeing on their shared domain). The algebra is **finitarily compatibly complete** provided it has joins of each finite set of pairwise-compatible elements (corresponding to being closed under finite unions of partial functions that agree wherever their domains overlap).

A **finitary compatible completion** of a difference–restriction algebra $\mathfrak{A}$ is an embedding $\iota\colon \mathfrak{A} \hookrightarrow \mathfrak{C}$ of $\{-, \triangleright\}$-algebras such that $\mathfrak{C}$ is a difference–restriction algebra and finitarily compatibly complete and $\iota[\mathfrak{A}]$ is finite-join dense in $\mathfrak{C}$ (i.e., each $c \in \mathfrak{C}$ is a a finite join of elements of $\iota[\mathfrak{A}]$).

Theorem 2. *For each difference–restriction algebra $\mathfrak{A}$, the homomorphism $\eta_{\mathfrak{A}}\colon \mathfrak{A} \to (G \circ F)(\mathfrak{A})$ is the finitary compatible completion of $\mathfrak{A}$, where η is the unit of the adjunction of Theorem 1.*

We write $\mathbf{C}_{\text{fin}}\mathbf{DRA}$ for the full subcategory of **DRA** consisting of the difference–restriction algebras that are finitarily compatibly complete. We write $\mathbf{Stone}^{+}\mathbf{Et}$ for the full subcategory of **HausEt** consisting of the $\pi\colon X \twoheadrightarrow X_0$ such that X is locally compact and zero-dimensional.

Proposition 1. *The adjunction restricts to a duality between* $\mathbf{C}_{\text{fin}}\mathbf{DRA}$ *and* $\mathbf{Stone}^{+}\mathbf{Et}$.

Adjunction for difference–restriction algebras with operators

An n-ary operation Ω on $\mathfrak{A}$ is **compatibility preserving** if whenever a_i, a'_i are compatible, for all i, we have that $\Omega(a_1, \ldots, a_n)$ and $\Omega(a'_1, \ldots, a'_n)$ are compatible, and Ω is an **operator** if it is *normal* ($\Omega(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$) and is *additive* (also known as *join preserving*) in each argument.

Let σ be a functional signature (disjoint from $\{-, \triangleright\}$).

The category $\mathbf{DRA}(\sigma)$ has *objects*: algebras of the signature $\{-, \triangleright\} \cup \sigma$ whose $\{-, \triangleright\}$-reduct is a difference–restriction algebra, and such that the symbols of σ are interpreted as compatibility preserving operators, and *morphisms*: homomorphisms of $(\{-, \triangleright\} \cup \sigma)$-algebras.

Let $\pi\colon X \twoheadrightarrow X_0$ be a Hausdorff étale space and R an $(n+1)$-ary relation on X. The **compatibility relation** $C \subseteq X \times X$ is given by xCy if and only if $\pi(x) = \pi(y) \implies x = y$. Then R has the **compatibility property** if given $x_1Cx'_1, \ldots, x_nCx'_n$ and $Rx_1 \ldots x_{n+1}$ and $Rx'_1 \ldots x'_{n+1}$, we have $x_{n+1}Cx'_{n+1}$.

Given subsets $S_1, \ldots, S_n$ of X, define $\Omega_R(S_1, \ldots, S_n)$ by $\Omega_R(S_1, \ldots, S_n) := \bigcup_{x_1 \in S_1, \ldots, x_n \in S_n} \{x_{n+1} \in X \mid Rx_1 \ldots x_{n+1}\}$. The relation R is **spectral** if whenever $S_1, \ldots, S_n \subseteq X$ are compact open sets, then $\Omega_R(S_1, \ldots, S_n)$ is a compact open set. The relation R is **tight** if, for each $x_1, \ldots, x_{n+1} \in X$, the condition $\forall S_1, \ldots, S_n$ compact and open $(x_1 \in S_n, \ldots, x_n \in S_n \implies x_{n+1} \in \Omega_R(S_1, \ldots, S_n))$ implies $Rx_1, \ldots, x_{n+1}$.

Take a partial function $\varphi\colon X \rightharpoonup Y$ and $(n+1)$-ary relations R_X and R_Y on X and Y. Then φ satisfies the **reverse forth condition** if whenever $R_X x_1 \ldots x_{n+1}$ and $\varphi(x_1), \ldots, \varphi(x_n)$ are defined, then $\varphi(x_{n+1})$ is defined and $R_Y \varphi(x_1) \ldots \varphi(x_{n+1})$. The partial map φ satisfies the **back condition** if whenever $\varphi(x_{n+1})$ is defined and $R_Y y_1 \ldots y_n \varphi(x_{n+1})$, then there exist $x_1, \ldots, x_n \in \mathrm{dom}(\varphi)$ such that $\varphi(x_1) = y_1, \ldots, \varphi(x_n) = y_n$ and $R_X x_1 \ldots x_{n+1}$.

Definition 2. The category $\mathbf{HausEt}(\sigma)$ has *objects*: the objects of **HausEt** equipped with, for each $\Omega \in \sigma$, an $(n+1)$-ary tight spectral relation R_Ω that has the compatibility property, where n is the arity of Ω, and *morphisms*: morphisms of **HausEt** that satisfy the reverse forth condition and the back condition with respect to R_Ω, for every $\Omega \in \sigma$.

Theorem 3. *There is an adjunction $F'\colon \mathbf{DRA}(\sigma) \dashv \mathbf{HausEt}(\sigma)^{\mathrm{op}} : G'$ that extends the adjunction $F \dashv G$ of Theorem 1 in the sense that the appropriate reducts of $F'(\mathfrak{A})$ and $G'(\pi\colon X \twoheadrightarrow X_0)$ equal $F(\mathfrak{A})$ and $G(\pi\colon X \twoheadrightarrow X_0)$, respectively.*

Corollary 1. *(i) For every algebra $\mathfrak{A}$ in $\mathbf{DRA}(\sigma)$, the embedding $\eta_{\mathfrak{A}}\colon \mathfrak{A} \hookrightarrow (G' \circ F')(\mathfrak{A})$ is the finitary compatible completion of $\mathfrak{A}$. (A finitary compatible completion in $\mathbf{DRA}(\sigma)$ should be a morphism of $\mathbf{DRA}(\sigma)$, i.e. also preserve the additional operators, as $\eta_{\mathfrak{A}}$ indeed does.)*

(ii) There is a duality between the categories $\mathbf{C}_{\mathrm{fin}}\mathbf{DRA}(\sigma)$ and $\mathbf{Stone}^{+}\mathbf{Et}(\sigma)^{\mathrm{op}}$.

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Generating and counting finite $\mathbf{FL}_{ew}$-chains

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Toward a more systematic analysis of the several variants of supporting algebras for various kinds of propositional and modal many-valued logics, *FL_{ew}-algebras* (*Full Lambek calculus with exchange and weakening*, see, e.g., [14]) were introduced to generalize the most common algebraic structures, such as *Gödel* algebras (G, for short) [3], *MV*-algebras [8] (MV) on which *Łukasiewicz* logic is based [21], *product* algebras (Π) [17], and *Heyting* algebras (H) that may provide an infinitely-valued interpretation of *intuitionistic* logic [11, 16, 19]. Each of these logics offers unique capabilities that have proven beneficial across various disciplines, including mathematics, computer science, and particularly artificial intelligence, where they enhance expressive power and decision-making processes; this is particularly true in the case of *modal* many-valued logics [12, 9], which have already been applied in different contexts but are just starting to be studied in depth.

The structure of FL_{ew}-algebras is of interest for both mathematicians and computer scientists; indeed, FL_{ew}-algebras are precisely *bounded integral commutative residuated lattices*. This means that an FL_{ew}-algebra $\boldsymbol{A}$ is *lattice ordered* by a partial ordering relation $\leqslant$, with a top (1) and a bottom (0) element. When the order is linear, we use the term *FL_{ew}-chain*. The additional structure that distinguishes FL_{ew}-algebras from common bounded lattices is given by another internal operation, usually denoted by $\cdot$, and assumed to be commutative, associative and having 1 as neutral element, sometimes referred to as *t-norm*, that is, such that $(\boldsymbol{A}, \cdot, 1)$ is a *monoid*; hence, we will often refer to the multiplication as the *monoidal operation*. Intuitively, the multiplication in an FL_{ew}-algebra generalizes the interpretation of the logical conjunction. Moreover, an FL_{ew}-algebra is assumed to have the *residuation property*, that is, it is assumed that for any elements $a, b \in \boldsymbol{A}$, there exists a unique maximal element x such that $a \cdot x \leq b$; this element is denoted by $a \to b$, and the implication operator $\to$ generalizes the logical implication. Most commonly used algebras in the field of fuzzy and many-valued logics are particular cases of some FL_{ew}-algebra $(\boldsymbol{A}, \cdot, \to, 1, 0)$; each specific case differs from the others in how the monoidal operation is defined.

While fuzzy logics are generally based on *infinite* algebras (typically built on the interval $[0, 1]$ of real numbers), the *finite* case is very interesting in practical cases [2]; among other contexts, datasets in machine learning are finite by definition, naturally leading to finite descriptions of patterns.

The question of probing a variety of finite algebras in order to count its non-isomorphic elements is a very natural one. De Baets and Mesiar [5] count the number of different t-norms that can be built on a chain of length n. Bartušek and Navara [6] solve the same problem by proposing a tool that actually generates all such t-norms. Belohlavek and Vychodil [7] again answer the question of generating all different residuated lattices, although, according to their definition, they actually focus on FL_{ew}-algebras of size n. Finally, Galatos and Jipsen [15] publish the set of all different FL_{ew}-algebras of size up to 6. Notwithstanding, the actual algorithm used for generation is published only in [7], and no database of FL_{ew}-algebras is actually current available for further analysis. Furthermore, no explicit bound for the number of different FL_{ew}-algebras has been given, and the numerical results are limited to the published constants.

In this work, we approach, again, the problem of counting and generating all different FL_{ew}-chains of size n, and, in particular: (i) we use a novel approach to this problem based on a topological interpretation of residuation theory, which shares some similarities with Scott's work in domain theory [22, 1]; (ii) we provide an explicit bound for the number of different FL_{ew}-chains of size n; (iii) we provide an accessible and open-source algorithmic tool for generating and counting FL_{ew}-chains as part of a long-term open-source framework for learning and reasoning, namely Sole.jl[1]. In particular, the tool can be found in the ManyValuedLogics submodule of the SoleLogics.jl[2] package, which provides the core data structures and functions for an easy manipulation of propositional, modal and many-valued logics. To ease the reader, the tool is also available in a standalone repository[3].

[1] https://github.com/aclai-lab/Sole.jl
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[3] https://github.com/aclai-lab/LATD2025a

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The Beth companion: making implicit operations explicit. Part II

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While arbitrary implicit operations are defined in terms of existential positive formulas, most familiar ones can be defined simply by conjunctions of equations. For instance,

1. the implicit operation of "taking the inverse y of x" is defined in monoids by
$$(x \cdot y \approx 1) \,\&\, (y \cdot x \approx 1);$$
2. that of "taking the complement y of x" is defined in bounded distributive lattices by
$$(x \wedge y \approx 0) \,\&\, (x \vee y \approx 1);$$
3. that of "taking the meet y of x_1 and x_2" is defined in Hilbert algebras by
$$(y \to x_1 \approx 1) \,\&\, (y \to x_2 \approx 1) \,\&\, (x_1 \to (x_2 \to y) \approx 1).$$

The following result explains, at least in part, why this is the case. To this end, given a class of algebras K and a term $t(x_1, \dots, x_n)$, we say that a first-order formula $\varphi(x_1, \dots, x_n, y)$ *defines* t in K when

$$\mathsf{K} \vDash t(x_1, \dots, x_n) \approx y \leftrightarrow \varphi(x_1, \dots, x_n, y).$$

Theorem 1. *Let K be a quasivariety with the amalgamation property and M a pp expansion of K. Then for each term $t(x_1, \dots, x_n)$ of M there exists a conjunction of equations $\varphi_t(x_1, \dots, x_n, y)$ in the language of K which defines t in M.*

The possibility of defining implicit operations in terms of conjunctions of equations (as opposed to arbitrary pp formulas) acquires special importance when applied to Beth companions.

Definition 1. Let K be a quasivariety. A class of algebras M is said to be a *Beth completion* of K when it is a Beth companion and for each term $t(x_1, \ldots, x_n)$ of M there exists a conjunction of equations $\varphi_t(x_1, \ldots, x_n, y)$ in the language of K which defines t in M.

For instance, Abelian groups, Boolean algebras, and implicative semilattices are Beth completions of the quasivarieties of commutative cancellative monoids, bounded distributive lattices, and Hilbert algebras, respectively.

In view of the following result, the Beth completion of a quasivariety K is unique (up to term-equivalence), when it exists. In this case, a class is a Beth completion of K iff it is a Beth companion of K.

Proposition 1. *Let K be a quasivariety with a Beth completion. Then every Beth companion of K is a Beth completion of K. Furthermore, all the Beth completions of K are term-equivalent.*

From Theorem 1 we obtain the following sufficient condition for a Beth companion to be a Beth completion.

Theorem 2. *Let K be a quasivariety with the amalgamation property. If K has a Beth companion, then it also has a Beth completion.*

Notably, Beth completions retain many interesting properties from their original quasivariety. The next observation captures a few of them.

Proposition 2. *The following conditions hold for a quasivariety K with a Beth completion M:*

1. *if K is a variety, then so is M;*
2. *every lattice equation valid in the lattices of K-congruences of the members of K is also valid in the lattices of M-congruences of the members of M.*

Our aim is to show that, not only do Beth completions retain some of the properties of their original quasivarieties, but that these properties are often significantly enhanced in Beth completions (see Theorem 3).

To this end, we recall that a variety is said to be *arithmetical* when it is both congruence distributive and congruence permutable. Furthermore, we recall that a member $\mathbf{A}$ of a quasivariety K is *relatively finitely subdirectly irreducible* (RFSI for short) when the identity congruence of $\mathbf{A}$ is meet-irreducible in the lattice $\mathsf{Con}_{\mathsf{K}}(\mathbf{A})$ of K-congruences of K. The class of RFSI members of K will be denoted by K_{RFSI}. When K is a variety, we drop the "relatively" and write simply K_{FSI}. Lastly, a variety is said to have the *congruence extension property* when for each $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$, every congruence of $\mathbf{A}$ can be extended to a congruence of $\mathbf{B}$.

Our main result takes the following form.

Theorem 3. *Let* K *be a relatively congruence distributive quasivariety such that* K_{RFSI} *is closed under nontrivial subalgebras. The following conditions hold for every Beth completion* M *of* K*:*

1. M *is a variety;*
2. M *is arithmetical;*
3. M *has the congruence extension property;*
4. M_{FSI} *is closed under nontrivial subalgebras.*

In other words, in the Beth completion we gain the congruence extension property, as well as the following improvements:

$$\text{quasivariety} \longmapsto \text{variety};$$
$$\text{relative congruence distributivity} \longmapsto \text{arithmeticity}.$$

At the same time, we preserve our assumptions on the class of RFSI algebras.

For instance, Theorem 3 provides a general explanation of why, moving from arbitrary bounded distributive lattices to Boolean algebras, we gain congruence permutability. For recall that the class of Boolean algebras is the Beth completion of the class of bounded distributive lattices which, in turn, is amenable to Theorem 3. In view of condition 2, Boolean algebras must be arithmetical and, therefore, congruence permutable.

Computational complexity of satisfiability problems in Łukasiewicz logic

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In dealing with optimization problems, a crucial role is played by the maximal satisfiability problem of the Boolean logic, henceforth denoted by MaxSAT_B, as well as its weighted version, denoted by $\mathsf{WMaxSAT}_B$, and the partial (weighted) satisfiability problem $\mathsf{P(W)MaxSAT}_B$. These problems are quite relevant since they can have theoretical as well as practical applications. Indeed, $\mathsf{WMaxSAT}_B$ and MaxSAT_B are among the first examples of $\mathsf{FP}^{\mathsf{NP}}$ and $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$ complete problems[1] (see [4]). Furthermore, one of the most commonly used strategies to prove the hardness of a given problem with respect to the aforementioned complexity classes is to find a metric reduction of it starting from $\mathsf{WMaxSAT}_B$ or MaxSAT_B. At the same time, many real-world problems can be encoded using the $\mathsf{PMaxSAT}_B$ framework (see, for example, [1] for an application to data analysis).

The first attempts to study the maximum satisfiability problem within the context of Łukasiewicz logic, denoted as MaxSAT, can be found in [5] and [3]. The main motivation for this generalization, as outlined in the introduction of [5], is that Łukasiewicz logic offers a richer framework, allowing certain problems to be naturally expressed in this language– for instance, problems involving continuous variables. In [3], the authors proved that MaxSAT is $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$-complete.

In many real-life applications, it is often beneficial to prioritize the satisfiability of certain formulas over others. This can be achieved by assigning to each formula φ a weight a_φ, denoting its relative importance. Furthermore, Łukasiewicz logic allows us to deal with intermediate truth values between absolute falsity and absolute truth. Putting these two remarks together, the first results we present in this talk is a *(weighted) version* of the maximum satisfiability problem in Łukasiewicz logic. Specifically, we say that a formula φ is r-satisfiable, with $r \in (0,1]$, if there exists a valuation v such that $v(\varphi) \geqslant r$. In the context of Łukasiewicz logic, it was proved in [6] that r-satisfiability is NP-complete. Furthermore, in the concluding remarks of [3], the authors defined the maximum r-satisfiability problem, $\mathsf{MaxSAT}_\mathsf{r}$, similarly to the MaxSAT problem, with the distinction that it focuses on the maximum number of formulas r-satisfied by a valuation. Hence,

[1] We recall that a function f belongs to the class $\mathsf{FP}^{\mathsf{NP}}$ if it is computable by a polynomial-bounded Turing machine with oracle NP. Similarly, f belongs to $\mathsf{FP}^{\mathsf{NP}}[O(\log(n))]$ if $f \in \mathsf{FP}^{\mathsf{NP}}$ and f is computable using at most $O(\log(n))$ queries.

we consider the following problem.

Definition 1. Let $r \in (0,1]$ be a rational number, and let F be a (non-empty) multiset of Łukasiewicz formulas. Let $0 \neq a_\varphi \in \mathbb{N}$ be the weight associated to the formula $\varphi \in F$. The $\mathsf{WMaxSAT_r}$ problem is the optimization problem that computes the maximum $0 \leqslant k \leqslant \sum_{\varphi \in F} a_\varphi$ such that there exists a valuation v such that $k = \sum_{\varphi \in S} a_\varphi$, where $S \subseteq F$ is the multiset of all formulas r-satisfied by v.

The problem $\mathsf{WMaxSAT}$ is obtained from Definition 1 by fixing $r = 1$. The problem $\mathsf{MaxSAT_r}$ is obtained from Definition 1 when all weights are equal to 1.

The first results we show are contained in next theorem.

Theorem 1. *The following hold.*

1. $\mathsf{WMaxSAT}$ *is* $\mathsf{FP^{NP}}$-complete.
2. $\mathsf{MaxSAT_r}$ *is* $\mathsf{FP^{NP}}[O(\log(n))]$-complete.
3. $\mathsf{WMaxSAT_r}$ *is* $\mathsf{FP^{NP}}$-complete.

Specifically, Theorem 1(1) is proved by reducing the (weighted) satisfiability in Łukasiewicz logic to a MIP problem, similarly to what is done in [2]; Theorem 1(3) is proved using a metric reduction to WMaxSAT. The proof of Theorem 1(2) is inspired by the results of [3], where the authors define the problem but leave open the task of finding its computational complexity.

To conclude this talk, we introduce the *partial (weighted) r-satisfiability problem*, denoted by $\mathsf{P(W)MaxSAT_r}$, in the context of Łukasiewicz logic. Formally, we consider the following problem.

Definition 2. Let $r \in (0,1]$ be a rational number, and let H and S be two multiset of Łukasiewicz formulas, with $S \neq \varnothing$. Let $0 \neq a_\varphi \in \mathbb{N}$ be the weight associated to the formula $\varphi \in S$. The $\mathsf{PWMaxSAT_r}$ problem is the optimization problem that computes the maximum $0 \leqslant k \leqslant \sum_{\varphi \in S} a_\varphi$ such that there exists a valuation v such that $v[H] = 1$ and $k = \sum_{\varphi \in T} a_\varphi$, where $T \subseteq S$ the multiset of formulas r-satisfied by v. If H is not satisfiable, by definition the solution of $\mathsf{PWMaxSAT_r}$ is $-\infty$.

The problem $\mathsf{PWMaxSAT}$ is obtained from Definition 2 by fixing $r = 1$, and the problem $\mathsf{PMaxSAT_r}$ is obtained by Definition 2 when $a_\varphi = 1$ for all $\varphi \in S$. The following result can be proved by generalizing the arguments used to prove Theorem 1.

Theorem 2. *The following hold.*

1. $\mathsf{PWMaxSAT}$ *is* $\mathsf{FP^{NP}}$-complete.

2. $\mathsf{PMaxSAT_r}$ *is* $\mathsf{FP^{NP}}[O(\log(n))]$-complete.

3. $\mathsf{PWMaxSAT_r}$ *is* $\mathsf{FP^{NP}}$-complete.

We remark that $\mathsf{PWMaxSAT_r}$, as it happens for its Boolean counterpart, has a lot of potential to real-world applications. Indeed, any continuous non-linear functions in n variables $f(x_1, \ldots, x_n) : [0,1]^n \to [0,1]$ can be approximated using rational piecewise linear functions. By the results of [7], such functions can be represented by a pair (H, φ), where $H \cup \{\varphi\}$ is a set of Łukasiewicz formulas in the propositional variables Var and $\{x_1, \ldots, x_n\} \subseteq \mathsf{Var}$. This representation has the property that, for any continuous and piecewise function f represented by (H, φ), if v is a $[0,1]$-valued evaluation that satisfies H, then $f(v(x_1), \ldots, v(x_n)) = v(\varphi)$. Hence, $\mathsf{PWMaxSAT_r}$ can be used as a potential framework for the resolution of many optimization problems.

Finally, if time allows it, we explore appropriate versions of the satisfiability problems outlined above within the context of fuzzy probabilistic logics.

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Bochvar algebras, Płonka sums, and twist products

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Abstract

The proper quasivariety $\mathscr{BCA}$ of Bochvar algebras, which serves as the equivalent algebraic semantics of Bochvar's external logic, was introduced by Finn and Grigolia in [6] and extensively studied in [4]. We show that the algebraic category of Bochvar algebras is equivalent to a category whose objects are pairs consisting of a Boolean algebra and a meet-subsemilattice (with unit) of the same. We also show that one of the functors that induce the equivalence can be equivalently defined either by means of a Płonka sum construction, or by means of a twist product construction.

In 1938, the Russian mathematician Dmitri Anatolyevich Bochvar published the influential paper "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus"[1], in which he introduced a 3-valued logic aimed at resolving set-theoretic and semantic paradoxes. His proposal diverged significantly from other related approaches in at least two key aspects. First and foremost, the third, non-classical value $\frac{1}{2}$ was *infectious* in any sentential compound involving the standard, or *internal*, propositional connectives $\neg, \wedge, \vee$. This means that a formula would be assigned the value $\frac{1}{2}$ iff at least one variable within it was assigned $\frac{1}{2}$. This third value was interpreted as "paradoxical". Second, the language of Bochvar's logic included *external* unary connectives J_0, J_1, J_2 (written in Finn and Grigolia's notation) that, unlike the internal connectives, could output only Boolean values.

Although the merits of Bochvar's logic as a solution to the paradoxes remain highly debatable, its influence on successive developments in 3-valued logic has been significant. The internal fragment of Bochvar's logic was characterised by Urquhart [6] through the imposition of a variable inclusion strainer on the consequence relation of classical propositional logic. Building on this result, a general framework for *right variable inclusion logics* has been proposed (see [1] for a detailed account). In this context, the celebrated algebraic construction of *Płonka sums* is extended from algebras to logical matrices. Specifically, each logic L is paired with a "right variable inclusion companion" L^r whose matrix models are decomposed as Płonka sums of models of L. Notably, Bochvar's internal logic serves as the right variable inclusion companion of classical logic.

Studies on Bochvar's external logic, by contrast, are comparatively scarce. Finn and Grigolia [6] provided an algebraic semantics for it with respect to the quasivariety of *Bochvar algebras*. However, their work does not employ the standard toolbox or terminology of abstract algebraic logic. Adopting a more mainstream approach, the papers [2, 4] extend Finn and Grigolia's completeness theorem to a full-fledged algebraisability result, and offer a representation of Bochvar algebras that refines the Płonka sum representations of their involutive bisemilattice reducts. We present Bochvar algebras in a simplified signature where the definable operation symbols J_0, J_1 are omitted.

Definition 1. A Bochvar algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, J_2, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ that satisfies the following identities:

1. $\varphi \vee \varphi \approx \varphi$;
2. $\varphi \vee \psi \approx \psi \vee \varphi$;
3. $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$;
4. $\varphi \wedge (\psi \vee \delta) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$;
5. $\neg(\neg\varphi) \approx \varphi$;
6. $\neg 1 \approx 0$;
7. $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$;
8. $0 \vee \varphi \approx \varphi$;
9. $J_2\neg J_2\varphi \approx \neg J_2\varphi$;
10. $J_2\varphi \approx \neg(J_2\neg\varphi \vee \neg(J_2\varphi \vee J_2\neg\varphi))$;
11. $J_2\varphi \vee \neg J_2\varphi \approx 1$;
12. $J_2(\varphi \vee \psi) \approx (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
13. $J_2\neg\varphi \approx J_2\neg\psi \ \&\ J_2\varphi \approx J_2\psi \ \Rightarrow\ \varphi \approx \psi$.

We show that the algebraic category of Bochvar algebras is equivalent to a category whose objects are pairs consisting of a Boolean algebra and a meet-subsemilattice (with unit) of the same. This equivalence instantiates the general theory of adjunctions between quasivarieties proposed by Moraschini [8].

Definition 2. A *Bochvar system* is a pair $\mathbb{B} = \langle \mathbf{B}, \mathbf{I}\rangle$ such that $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\mathbf{I} = \langle I, \wedge, 1\rangle$ is a meet-subsemilattice with unit of $\mathbf{B}$.

Let $\mathfrak{B}$ denote the algebraic category of Bochvar algebras. We now define a category $\mathfrak{S}$ whose objects are Bochvar systems. If $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{I}_1 \rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{I}_2 \rangle$ are objects in $\mathfrak{S}$, a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$ is a homomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in I_2$ for every $i \in I_1$. Observe that any such g is also a homomorphism from $\mathbf{I}_1$ to $\mathbf{I}_2$.

Theorem 1. *The categories $\mathfrak{B}$ and $\mathfrak{S}$ are equivalent.*

Proof. (Sketch.) Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ be a Bochvar system. We define

$$\mathbb{A}_{\mathbb{B}} = \left\langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}^{\partial}, \{p_{ij} : i \leqslant_{\mathbf{I}^{\partial}} j\} \right\rangle$$

such that:

- for all $i \in I$, $\mathbf{A}_i := \mathbf{B}/[i)$;
- $\mathbf{I}^{\partial}$ is the lower-bounded join-semilattice dual to $\mathbf{I}$;
- for all $i, j \in I$ such that $i \leqslant_{\mathbf{I}^{\partial}} j$, $p_{ij}(a/[i)) := (a/[i))/[j)$.

$\mathbb{A}_{\mathbb{B}}$ is a semilattice direct system of Boolean algebras, whence the Płonka sum $\mathscr{P}_{\nmid}(\mathbf{A}_i)_{i \in I}$ over it is an involutive bisemilattice [1, Ch. 2]. By the results in [4], this is the underlying involutive bisemilattice of a unique Bochvar algebra, noted $\mathbf{A}_{\mathbb{B}}$.

For the other direction, let

$$\mathbf{A} = \langle A, \wedge, \vee, \neg, J_2, 0, 1 \rangle$$

be a Bochvar algebra, whose involutive bisemilattice reduct decomposes as $\mathscr{P}_{\nmid}(\mathbf{A}_i)_{i \in I}$. We define $\mathbb{B}_{\mathbf{A}} := \langle \mathbf{A}_{i_0}, \mathbf{K} \rangle$, where $K = \{J_2^{\mathbf{A}}(1^{A_i}) : i \in I\}$, and for $J_2^{\mathbf{A}}(1^{A_i}), J_2^{\mathbf{A}}(1^{A_j}) \in K$, $J_2^{\mathbf{A}}(1^{A_i}) \leqslant_{\mathbf{K}} J_2^{\mathbf{A}}(1^{A_j})$ iff $j \leqslant_{\mathbf{I}} i$. We have that $\mathbb{B}_{\mathbf{A}}$ is a Bochvar system.

We now define the map Γ as follows:

- If $\mathbf{A}$ is an object in $\mathfrak{B}$, let $\Gamma(\mathbf{A}) := \mathbb{B}_{\mathbf{A}}$.
- If $f : \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in $\mathfrak{B}$, let $\Gamma(f)$ be the restriction of f to $\mathbf{A}_{1_{i_0}}$.

Similarly, we define the map Ξ as follows:

- If $\mathbb{B}$ is an object in $\mathfrak{S}$, let $\Xi(\mathbb{B}) := \mathbf{A}_{\mathbb{B}}$.
- If $g : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathfrak{S}$, let $\Xi(g)$ be defined as follows: $\Xi(g)(a/[i)) := g(a)/[g(i))$.

Γ and Ξ are functors that induce an equivalence between $\mathfrak{B}$ and $\mathfrak{S}$. □

Interestingly, the functor Ξ can be equivalently defined by resorting not to a Płonka-type construction, but rather to the definition of a *twist product algebra*.

Definition 3. Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ be a Bochvar system. The *twist product algebra* over $\mathbb{B}$ is the algebra

$$Tw(\mathbb{B}) = \langle T, \wedge^{Tw(\mathbb{B})}, \vee^{Tw(\mathbb{B})}, \neg^{Tw(\mathbb{B})}, J_2^{Tw(\mathbb{B})}, 0^{Tw(\mathbb{B})}, 1^{Tw(\mathbb{B})} \rangle$$

of type $\langle 2, 2, 1, 1, 0, 0 \rangle$, such that (omitting superscripts when denoting the operations in $\mathbf{B}$):

- $T := \{\langle a, b \rangle : a, b \in B, a \wedge b = 0, a \vee b \in I\}$;
- $\langle a, b \rangle \wedge^{Tw(\mathbb{B})} \langle c, d \rangle := \langle a \wedge c, (b \wedge d) \vee (b \wedge c) \vee (a \wedge d) \rangle$;
- $\langle a, b \rangle \vee^{Tw(\mathbb{B})} \langle c, d \rangle := \langle (a \wedge c) \vee (a \wedge d) \vee (b \wedge c), b \wedge d \rangle$;
- $\neg^{Tw(\mathbb{B})} \langle a, b \rangle := \langle b, a \rangle$;
- $J_2^{Tw(\mathbb{B})} \langle a, b \rangle := \langle a, \neg a \rangle$;
- $0^{Tw(\mathbb{B})} := \langle 0, 1 \rangle$;
- $1^{Tw(\mathbb{B})} := \langle 1, 0 \rangle$.

Theorem 2. *$Tw(\mathbb{B})$ is a Bochvar algebra that is isomorphic to $\mathbf{A}_{\mathbb{B}}$.*

This observation might point both to a possible extension of the theory of twist products beyond the lattice-ordered case, and to a further exploration of the relationships between the constructions of Płonka sums and twist products. Moreover, it might relate to recent work on twist constructions and residuated lattices with conuclei [5, 9].

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Modal $\mathbf{FL}_{ew}$-algebra satisfiability through first-order translation

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Modal logics offer a valid treatment for temporal and spatial data, which are critical in modeling many real-world scenarios and, therefore, are becoming more popular by the day in artificial intelligence applications, specifically when dealing with symbolic machine learning. Some notable examples are [18, 20], introducing modal logics for treating interval temporal relations and topological (i.e., spatial) relations, respectively. However, practitioners handling temporal and spatial data typically encounter challenges, as sensing and discretizing signals that often introduce inaccuracies in the data. Fuzzy logics are renowned as a common approach to deal with uncertainty and unclear boundaries in the data. Furthermore, Melvin Fitting proposed in [12] a many-valued approach leveraging Heyting algebras to tackle many-expert scenarios, another compelling application in artificial intelligence. In this talk, we want to present a framework that is general enough to treat modal many-valued logics, including Fitting's proposal, and can be endowed with reasoning tools suitable for real-world applications.

FL_{ew}-algebras (*Full Lambek calculus with exchange and weakening*, see, e.g., [14]) proved to be a valid candidate, as it generalizes most common algebraic structures of many-valued logics, such as *Gödel* algebras (G, for short) [3], *MV*-algebras [8]

(MV) on which *Łukasiewicz* logic is based [21], *product* algebras (Π) [17], and *Heyting* algebras (H). FL_{ew}-algebras are *bounded integral commutative residuated lattices*; i.e., an FL_{ew}-algebra $\boldsymbol{A}$ is a *lattice* ordered by a partial ordering relation $\leqslant$, with a top (1) and a bottom (0) element. When the order is linear, we use the term *FL_{ew}-chain*. The difference between FL_{ew}-algebras and common bounded lattices is the presence of an internal operation, usually denoted by $\cdot$, and assumed to be commutative, associative and having 1 as a neutral element, usually referred to as *t-norm*, that is, such that $(\boldsymbol{A}, \cdot, 1)$ is a *monoid*; hence, we will often refer to the multiplication as the *monoidal operation*. Intuitively, the multiplication in an FL_{ew}-algebra generalizes the interpretation of the logical conjunction. Moreover, an FL_{ew}-algebra is assumed to have the *residuation property*, that is, it is assumed that for fixed elements $a, b \in \boldsymbol{A}$, there exists a unique maximal element x such that $a \cdot x \leq b$; this element is denoted by $a \to b$, and the implication operator $\to$ generalizes the logical implication. All commonly used algebras in the field of many-valued logics are particular cases of some FL_{ew}-algebra $(\boldsymbol{A}, \cdot, \to, 1, 0)$; each specific case differs from the others in how the monoidal operation is defined.

While *modal* many-valued logics [12, 9] have already been applied in different contexts, they are just starting to be studied in depth. Automated theorem proving for modal FL_{ew}-algebra formulas encompassing a many-valued generalization of Halpern and Shoham's interval temporal logic [18] has been tackled in [4] using a tableaux system inspired by the one proposed by Melvin Fitting in [13] and already extended to Heyting Algebras in [10]. When tackling formulas satisfiability and validity in FL_{ew}-algebras (and, more generally, many-valued logics defined over a lattice representing a partial order), the problem can be relaxed to finding, given a formula φ and a value α in the algebra, if a model exists such that (resp., for all possible models) the formula has at least value α. This problem is referred to α-satisfiability (resp. α-validity).

In this work, we propose a different approach leveraging well-known sat and smt solvers, such as $z3$, with the hope of gaining better performance while maintaining some sort of interpretability. In order to do so, one has to translate the α-satisfiability problem to a two-sorted first-order problem, with a first sort $\mathscr{A}$ representing the values in the FL_{ew}-algebra and a second sort $\mathscr{W}$ representing the worlds, such that given a formula φ interpreted on an FL_{ew}-algebra A, φ is α-satisfiable if and only if it exists $\mathscr{M}, w \in \mathscr{W}$ so that $\mathscr{V}_{\mathscr{M}}(w, a) \succeq \alpha$.

We provide an accessible and open-source algorithmic tool for (i) defining finite FL_{ew}-algebras, (ii) writing formulas in a specified FL_{ew}-algebra, and (iii) asking α-satisfiability for a given value α in the algebra of the formula through a first-order translation and making use of a sat or a smt solver, such as $z3$. This tool is offered as part of a long-term open-source framework for learning and reasoning, namely Sole.jl[1]. In particular, the tool can be found in the ManyValuedLogics submodule of the SoleLogics.jl[2] package, which provides the core data structures and functions for an easy manipulation of propositional, modal and many-valued

[1] https://github.com/aclai-lab/Sole.jl

[2] https://github.com/aclai-lab/SoleLogics.jl

logics. For the benefit the reader, the tool is also available in a standalone repository[3], using many-valued Halpern and Shoham's interval temporal logic as an example.

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On split extensions of hoops

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Internal actions have been introduced in [3] by F. Borceux, G. Janelidze, and G. M. Kelly as a means to generalize the connection between actions and split extensions from groups and Lie algebras to arbitrary semi-abelian categories. However, in certain settings such as Orzech categories of interest [20] internal actions are often expressed in terms of external actions, i.e., via a set of maps which satisfy a certain set of identities. In this talk, we are gonna study external actions and split extensions in the category Hoops of hoops, with a focus on those split extensions which strongly splits. In particular, we say that a split extension

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

strongly splits, or has a *strong section*, if the morphism s is a *strong section* of p, i.e. if the equation

$$a \to s(b) = sp(a) \to s(b)$$

holds for every $a \in A$ and $b \in B$.

When a split extension

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

in Hoops strongly splits, then the semidirect product $X \ltimes_\xi B$ of X and B with respect to the corresponding internal action ξ is given by the set

$$\{(x, b) \in X \times B \mid \; s(b) \to (s(b) \cdot x) = x\}$$

together with the operations

$$(x, b) \to (y, b') = (s(b' \to b) \to (x \to y),\; b \to b'),$$

$$(x, b) \cdot (y, b') = (s(b \cdot b') \to (s(b \cdot b') \cdot x \cdot y),\; b \cdot b')$$

and

$$1_{X \ltimes_\xi B} = (1, 1, 1).$$

Split extensions with strong section in the category Hoops can be described in terms of *strong external actions* [19], i.e., a pair of maps

$$f \colon B \times X \to X \colon (b, x) \mapsto f_b(x),$$

$$g \colon B \times X \to X \colon (b, x) \mapsto g_b(x)$$

such that

E1. $f_b(1) = g_b(1) = 1$;

E2. $f_1 = g_1 = \mathrm{id}_X$;

E3. $f_{b_1 \cdot b_2}(x \cdot g_{b_1}(x \to y)) = f_{b_1 \cdot b_2}(x \cdot (x \to y))$;

E4.

$$g_{(b_3 \to (b_1 \cdot b_2))}(f_{b_1 \cdot b_2}(x \cdot y) \to z) = \\ = g_{(b_2 \to b_3) \to b_1}(x \to g_{b_3 \to b_2}(y \to z));$$

for any $b, b_1, b_2, b_3 \in B$ and $x, y, z \in X$.

In particular, there is a bijection τ_B between the set $\mathrm{EAct}_{\mathrm{ss}}(B, X)$ of strong external actions of B on X and the set $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ of isomorphism classes of split extensions of B by X that strongly splits. Indeed, we can define τ_B as the map that sends every split extension in Hoops that strongly splits

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B,$$

to the pair of maps $f, g \colon B \times X \to X$ defined by

$$f_b(x) = s(b) \to (s(b) \cdot x) \quad \text{and} \quad g_b(x) = s(b) \to x.$$

It is possibile to show that (f, g) defines a strong external action of B on X. Moreover, the map μ_B which sends a strong external action $f, g \colon B \times X \to X$ to the split extension of B by X

$$X \xrightarrow{\iota_1} Y \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} B$$

where

$$Y = \{(x, b) \in X \times B | \ f_b(x) = x\}$$

and

$$(x, b) \to (y, b') = (g_{b' \to b}(x \to y),\ b \to b'),$$
$$(x, b) \cdot (y, b') = (f_{b \cdot b'}(x \cdot y),\ b \cdot b')$$

is the inverse of the map τ_B.

As a consequence, there is a natural isomorphism

$$\tau \colon \ \mathrm{SplExt}_{\mathrm{ss}}(-, X) \cong \mathrm{EAct}_{\mathrm{ss}}(-, X),$$

where $\mathrm{SplExt}_{\mathrm{ss}}(-, X) \colon \mathsf{Hoops}^{\mathrm{op}} \to \mathsf{Set}$ is the functor which assigns to any hoop B, the set $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ and

$$\mathrm{EAct}_{\mathrm{ss}}(-, X) \colon \mathsf{Hoops}^{\mathrm{op}} \to \mathsf{Set}$$

is the functor which maps every hoop B to $\mathrm{EAct}_{\mathrm{ss}}(B, X)$.

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Uniform validity of atomic Kreisel-Putnam rule in monotonic proof-theoretic semantics

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In recent years, many completeness or incompleteness results have been proved for variants of *monotonic proof-theoretic semantics* (mPTS), see [1, 2, 3, 7, 8, 9, 13, 12, 14, 15, 16]. Below, I shall understand a model for mPTS as a set $\mathfrak{B}$ of *atomic rules R of level $n \geqslant 0$* of the form

$$\dfrac{\begin{matrix}[\Re_1] & & [\Re_n] \\ A_1 & \ldots & A_n\end{matrix}}{A}\, R$$

where A_i, A are atoms from the underlying language, and $\Re_i$ is an atomic rule of level $n-2$ discharged by R ($i \leqslant n$). That A is *a consequence of* Γ *on* $\mathfrak{B}$, written $\Gamma \models_{\mathfrak{B}} A$, is defined thus—limiting ourselves to propositional logic:

Definition 1. $\Gamma \models_{\mathfrak{B}} A \Longleftrightarrow$

- $\Gamma = \varnothing \Longrightarrow$
  - A is atomic $\Longrightarrow \vdash_{\mathfrak{B}} A$
  - $A = B \wedge C \Longrightarrow \models_{\mathfrak{B}} B$ and $\models_{\mathfrak{B}} C$
  - $A = B \vee C \Longrightarrow \models_{\mathfrak{B}} B$ or $\models_{\mathfrak{B}} C$
  - $A = B \rightarrow C \Longrightarrow B \models_{\mathfrak{B}} C$
- $\Gamma \neq \varnothing \Longrightarrow \forall \mathfrak{C} \supseteq \mathfrak{B}\ (\models_{\mathfrak{C}} \Gamma \Longrightarrow\ \models_{\mathfrak{C}} A)$.

Definition 2. $\Gamma \models A \Longleftrightarrow \forall \mathfrak{B}\ (\Gamma \models_{\mathfrak{B}} A)$.

—for approaches where one also takes care of constraints on the specific kinds of (sets of) sets of atomic rules of given levels, see [9, 13, 14, 15].

These variants of mPTS have been qualified as *sentential*—see [13]—since they start with a primitive notion of consequence on $\mathfrak{B}$, and so differ from Prawitz's original approach [10, 11], where *consequence of A from Γ on $\mathfrak{B}$* is *defined* as existence of an argument from Γ to A valid on $\mathfrak{B}$—logical consequence is likewise defined as existence of a logically valid argument from Γ to A. Valid arguments on $\mathfrak{B}$ are here prior, and defined as follows (some preliminary definitions are required).

Definition 3. An *argument structure* is a tree $\mathcal{D}$ with nodes labelled by formulas, and edges standing for arbitrary inferences, which may discharge assumptions or atomic rules. The leaves are the *assumptions of* $\mathcal{D}$, while the root is the *conclusion of* $\mathcal{D}$.

Definition 4. $\mathcal{D}$ is *closed* if all its assumptions are discharged, and it is *open* otherwise.

Definition 5. $\mathcal{D}$ is canonical if it ends by applying an introduction, and it is *non-canonical* otherwise.

Definition 6. An *inference rule* R is a set of argument structures.

I write $\mathcal{D}[\mathcal{D}^{**}/\mathcal{D}^{*}]$ the result of replacing by $\mathcal{D}^{**}$ the sub-structure $\mathcal{D}^{*}$ of $\mathcal{D}$.

Definition 7. Let σ be a function from formulas A to (closed) argument structures with conclusion A. The *(closed) σ-instance* $\mathcal{D}^{\sigma}$ of $\mathcal{D}$ is $\mathcal{D}[\sigma(A_1), ..., \sigma(A_n)/A_1, ..., A_n]$, where $A_1, ..., A_n$ are the undischarged assumptions of $\mathcal{D}$.

Definition 8. A *reduction for* R is a function φ from argument structures to argument structures, defined on some $\mathbb{D} \subseteq R$ and such that, $\forall \mathcal{D} \in \mathbb{D}$,

- $\mathcal{D}$ is from Γ to $A \Longrightarrow \varphi(\mathcal{D})$ is from $\Gamma^{*} \subseteq \Gamma$ to A
- $\forall \sigma$, φ is defined on $\mathcal{D}^{\sigma}$, and $\varphi(\mathcal{D}^{\sigma}) = \varphi(\mathcal{D})^{\sigma}$.

Definition 9. Let $\mathfrak{J}$ be a set of reductions. $\mathcal{D}$ *reduces to* $\mathcal{D}^{*}$ *relative to* $\mathfrak{J}$, written $\mathcal{D} \leqslant_{\mathfrak{J}} \mathcal{D}^{*}$, if there is a sequence $\mathcal{D}_1, ..., \mathcal{D}_n$ such that $\mathcal{D} = \mathcal{D}_1$, $\mathcal{D}^{*} = \mathcal{D}_n$ and, $\forall i \leqslant n$, $\mathcal{D}_{i+1} = \mathcal{D}_i[\varphi(\mathcal{D}^{**})/\mathcal{D}^{**}]$ with $\varphi \in \mathfrak{J}$.

Definition 10. $\langle \mathcal{D}, \mathfrak{J} \rangle$ is *valid on* $\mathfrak{B} \Longleftrightarrow$

- $\mathcal{D}$ is closed $\Longrightarrow$
  - the conclusion A of $\mathcal{D}$ is atomic $\Longrightarrow \mathcal{D} \leqslant_{\mathfrak{J}} \mathcal{D}^{*}$ with $\mathcal{D}^{*}$ closed derivation of A in $\mathfrak{B}$
  - $\mathcal{D}$ is canonical $\Longrightarrow$ the immediate sub-structures of $\mathcal{D}$ are valid on $\mathfrak{B}$ when paired with $\mathfrak{J}$
  - $\mathcal{D}$ is non-canonical $\Longrightarrow \mathcal{D} \leqslant_{\mathfrak{J}} \mathcal{D}^{*}$ with $\mathcal{D}^{*}$ canonical valid on $\mathfrak{B}$ when paired with $\mathfrak{J}$
- $\mathcal{D}$ is open with assumptions $A_1, ..., A_n \Longrightarrow \forall \mathfrak{C} \supseteq \mathfrak{B}, \mathfrak{H} \supseteq \mathfrak{J}, \sigma$, $(\langle \sigma(A_i), \mathfrak{H} \rangle$ valid on $\mathfrak{C} \Longrightarrow \langle \mathcal{D}^{\sigma}, \mathfrak{H} \rangle$ valid on $\mathfrak{C})$.

Definition 11. $\langle \mathcal{D}, \mathfrak{J} \rangle$ is *logically valid* $\Longleftrightarrow$ it is valid on every $\mathfrak{B}$.

One might wonder whether (logical) consequence in sentential mPTS implies (logical) consequence in Prawitz's original approach, and vice versa. When the notion of set of reductions is liberal enough, then the two approaches do coincide, whence intuitionistic logic is incomplete relative to a Prawitzian-oriented framework with liberal sets of reductions [6].

In a stricter reading, sets of reductions are *base-independent* constructive functions for the uniform rewriting of argument structures. It is unclear how base-independence should be defined more precisely, but the rough idea is that sets of reductions are constructive (or possibly finite) sets of operations whose output values are described by relying only on structural properties of the input values and, in particular, with no reference to specific atomic bases and structures which such values reduce to on those bases—this seems to be how Prawitz thinks of reductions in [10, 11], but see also [5, 13]. Following [13], I shall call *uniform monotonic proof-theoretic semantics* (umPTS) this approach.

It is not clear how to attain a direct translation from mPTS to umPTS, mainly due to the fact that it is not clear how to prove in umPTS the mPTS clause for the open consequence case, i.e., with $\Gamma \neq \varnothing$,

$$\Gamma \models_{\mathfrak{B}} A \Longleftrightarrow \forall \mathfrak{C} \supseteq \mathfrak{B} \ (\models_{\mathfrak{C}} \Gamma \Longrightarrow \models_{\mathfrak{C}} A).$$

While the $\Longrightarrow$ direction of this clause trivially holds in umPTS too, the $\Longleftarrow$ might fail. The fact that the existence of closed $\langle \mathcal{D}, \mathfrak{J} \rangle$-s for the elements of Γ valid on $\mathfrak{C}$ implies the existence of a closed $\langle \mathcal{D}^*, \mathfrak{H} \rangle$ for A also valid on $\mathfrak{C}$, may not imply in any straightforward way the existence of an open $\langle \mathcal{D}^{**}, \mathfrak{K} \rangle$ from Γ to A valid on $\mathfrak{B}$, when $\mathfrak{K}$ is constrained to be uniform.

In my talk, however, I show that this issue can be overcome, and that incompleteness of intuitionistic logic with respect to umPTS can be actually proved without any need of translating results from mPTS to umPTS. This is done along the lines of some incompleteness proofs given in [1, 2, 8], which apply to a framework where the notion of validity is defined in terms of (intuitionistic) constructions.

As said, sets of reductions in umPTS are partial constructive functions of a special kind, so proofs referred to a notion of validity defined in terms of (intuitionistic) constructions may not be automatically transferable to umPTS. A rough attempt at turning the kind of constructions used in the incompleteness proofs from [1, 2, 8] into sets of reductions in umPTS, seems in fact to suggest that the sets of reductions thereby obtained would not be base-independent.

On the other hand, Pezlar [4] has recently proved that a generalised Kreisel-Putnam rule—where the antecedent of the premise is a Harrop formula—can be constructively justified through a selector with schematic elimination and equality type-theoretic rules. The formulation of Pezlar's selector in a Natural Deduction formalism seems to hint at the fact that something similar might be done for umPTS and, hence, that an incompleteness proof along the lines of [1, 2, 8] can be found also for Prawitz's original framework—monotonically understood. Indeed, this holds for the intuitionistically underivable Kreisel-Putnam rule restricted to atoms, i.e.,

$$\dfrac{p \to q \vee r}{(p \to q) \vee (p \to r)}\ \texttt{KP}a$$

by exhibiting a set of reductions for this rule which is clearly base-independent—however the concept of base-independence is defined—so the following is provable.

Theorem 1. *There is a base-independent set of reductions $\mathfrak{J}$ such that the open argument structure* KP*a is valid on every $\mathfrak{B}$.*

The required set of reductions is made up of the five functions $\varphi_1, \varphi_{2,i}, \varphi_{3,i}$ defined as follows.

$$\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathfrak{D}\\ B \vee C\end{matrix}}{A \to B \vee C}\to_I, 1}{(A \to B) \vee (A \to C)} \quad \overset{\varphi_1}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{matrix}\overline{\ A\ }\\ \mathfrak{D}\\ B \vee C\end{matrix}}{A \to B \vee C}\to_I}{(A \to B) \vee (A \to C)}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\overline{\ A\ }\\ \mathfrak{D}\\ B_i\end{matrix}}{B_1 \vee B_2}\vee_{I,i}}{A \to B_1 \vee B_2}\to_I}{(A \to B_1) \vee (A \to B_2)} \quad \overset{\varphi_{2,i}}{\Longrightarrow} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathfrak{D}\\ B_i\end{matrix}}{B_1 \vee B_2}\vee_{I,i}}{A \to B_1 \vee B_2}\to_I, 1}{(A \to B_1) \vee (A \to B_2)}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathfrak{D}\\ B_i\end{matrix}}{B_1 \vee B_2}\vee_{I,i}}{A \to B_1 \vee B_2}\to_I, 1}{(A \to B_1) \vee (A \to B_2)} \quad \overset{\varphi_{3,i}}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathfrak{D}\\ B_i\end{matrix}}{A \to B_i}\to_I, 1}{(A \to B_1) \vee (A \to B_2)}\vee_{I,i}$$

where $i = 1, 2$. The key observation for Theorem 1 is that reduction sequences out of a base-independent $\mathfrak{J}$ are just syntactic manipulations of argument structures, thus they go through over all bases, independently of whether the input argument structures are valid relative to $\mathfrak{J}$ on given (extensions of) atomic bases. In other words, when saying that non-canonical $\mathfrak{D}$ is valid relative to $\mathfrak{J}$ over $\mathfrak{B}$, we should read as independent of each other the two conjuncts which this condition amounts to via Definition 10, namely (every closed instance of) $\mathfrak{D}$ reduces relative to $\mathfrak{J}$ to a canonical argument structure $\mathfrak{D}^*$

and

the immediate sub-structures of $\mathfrak{D}^*$ are valid relative to $\mathfrak{B}$ when paired with $\mathfrak{J}$.

From Theorem 1 we immediately draw the following conclusion.

Corollary 1. *Intuitionistic logic is incomplete over* umPTS.

When Prawitz's completeness conjecture from [11] is formulated in a monotonic approach, umPTS is the kind of proof-theoretic semantics which the conjecture should refer to, whence Corollary 1 implies a refutation of the conjecture for this "orthodox" Prawitzian framework. Of course, one could require additional constraints on the notion of logical validity of arguments, e.g. some closure condition like

$$\langle \mathfrak{D}, \mathfrak{J} \rangle \iff \forall \star, \langle \star(\mathfrak{D}), \mathfrak{J}^\star \rangle \text{ is valid on every } \mathfrak{B}$$

where $\star$ is a function defined on formulas A that replaces atomic sub-formulas of A by formulas, $\star(\mathfrak{D})$ is the result of replacing each formula occurrence of A in $\mathfrak{D}$ by $\star(A)$, and $\mathfrak{J}^\star$ is a set of reductions such that, for every $\varphi \in \mathfrak{J}$, there is $\varphi^\star \in \mathfrak{J}^\star$ such that, when φ is defined on $\mathfrak{D}^*$ and $\varphi(\mathfrak{D}^*) = \mathfrak{D}^{**}$, then $\varphi^\star$ is defined on $\star(\mathfrak{D}^*)$ and $\varphi^\star(\star(\mathfrak{D}^*)) = \star(\mathfrak{D}^{**})$—observe that it may well be that $\mathfrak{J} = \mathfrak{J}^\star$. However, this is not the kind of logical validity which Prawitz refers his conjecture to in [11].

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Hölder's theorem for totally ordered monoids

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Hölder's theorem [3, 5, 6], one of the early classical results about ordered groups, states that a totally ordered group embeds into the additive ordered group of reals $\mathbb{R}$ if and only if it is Archimedean (informally speaking, if and only if it lacks infinitesimal elements). The ordered group $\mathbb{R}$ which features in Hölder's theorem lives in the variety of lattice-ordered groups, which is one of the most prominent varieties of residuated lattices. A fruitful research programme in this area has been to extend results about lattice-ordered groups to wider classes of residuated lattices [1]. Two important classes for this purpose are the variety of GBL-algebras and its subvariety of GMV-algebras [2]. These significantly extend the variety of lattice-ordered groups while still preserving some group-like behavior. In particular, in their study of the Archimedean property in residuated lattices, Ledda, Paoli and Tsinakis [4] recently extended Hölder's theorem to GBL-algebras, characterizing the subalgebras of the GMV-algebras $\mathbb{R}$, $\mathbb{R}^-$, and $[0,1]$ as precisely the strongly simple GBL-algebras (see below for more details). In the present work, we further extend Hölder's theorem beyond the residuated setting, obtaining a result in the spirit of [4] for totally ordered monoids which gives an abstract characterization of the *dense* subalgebras of the totally ordered monoids $\mathbb{R}$, $\mathbb{R}^-$, and $[0,1]$.

Some definitions will be needed to state these results. A *lattice-ordered monoid*, or *ℓ-monoid* for short, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, 1\rangle$ which is both a lattice and a monoid such that products distribute over binary meets and binary joins. An ℓ-monoid is *integral* if the monoidal unit 1 is the top element of the lattice reduct. A *totally ordered monoid*, or *tomonoid* for short, is an ℓ-monoid whose lattice reduct is totally ordered. A *residuated lattice* is an ℓ-monoid equipped with binary operations $\backslash$ and $/$ such that

$$y \leqslant x\backslash z \iff x \cdot y \leqslant z \iff x \leqslant z/y.$$

A *GBL-algebra* is a residuated lattice satisfying the divisibility equations

$$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x.$$

A *GMV-algebra* is a residuated lattice satisfying the stronger equations

$$x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x.$$

The variety of GMV-algebras in particular subsumes the varieties of ℓ-groups and MV-algebras. Key examples of commutative GMV-algebras are the additive ℓ-group of the reals $\mathbb{R}$, its negative cone $\mathbb{R}^-$ (an integral cancellative residuated lattice), and the standard MV-chain $[0, 1]$. These GMV-algebras have some important subalgebras: the additive ℓ-group of the integers $\mathbb{Z}$, its negative cone $\mathbb{Z}^-$, and the subalgebras $Ł_n$ of $[0, 1]$ with the universes $\{0/n, 1/n, \ldots, n/n\}$ for $n \geqslant 1$. We use $Ł_0$ to denote the trivial algebra.

A residuated lattice is *strongly simple* if it has no non-trivial proper convex subalgebras. It is *strongly semisimple* if $\{1\}$ is the intersection of all maximal proper convex subalgebras. A commutative residuated lattice is strongly (semi)simple if and only if it is (semi)simple in the universal algebraic sense.

Hölder's theorem for GBL-algebras ([4, Theorem 5.6])**.** A GBL-algebra is strongly simple if and only if it is isomorphic to one of the following:

(i) a subalgebra of $\mathbb{R}$,

(ii) a subalgebra of $\mathbb{R}^-$,

(iii) a subalgebra of $[0, 1]$.

In particular, each strongly simple GBL-algebra is commutative.

We now aim to extend this theorem to totally ordered monoids which are not necessarily residuated. This will require two modifications.

Firstly, the lattice of convex subalgebra is really a stand-in for the lattice of left congruences, or equivalently for the lattice of right congruences. A *left congruence* of an ℓ-monoid $\mathbf{L}$ is a lattice congruence θ such that $\langle a, b\rangle \in \theta$ implies $\langle ca, cb\rangle \in \theta$, and a *right congruence* is a lattice congruence θ such that $\langle a, b\rangle \in \theta$ implies $\langle ac, bc\rangle \in \theta$. A left congruence of a residuated lattice moreover satisfies the condition that $\langle a, b\rangle \in \theta$ implies $\langle c\backslash a, c\backslash b\rangle \in \theta$, while a right congruence satisfies the condition that $\langle a, b\rangle \in \theta$ implies $\langle a/c, b/c\rangle \in \theta$. In a residuated lattice, the lattices of left congruences, of right congruences, and of convex subalgebras are isomorphic. Beyond the residuated case, we need to explicitly work with the lattices of left and right congruences.

Secondly, as a result of dropping residuation from the signature, we now have more (left and right) congruences. The strongly simple residuated lattices $\mathbb{R}$, $\mathbb{R}^-$, and $[0, 1]$ have no residuated lattice congruences besides the identity and the total congruence. In contrast, for each non-empty downset I of $[0, 1]$ there is an ℓ-monoidal congruence $\Theta(I)$ such that $\langle a, b\rangle \in \Theta(I)$ if and only if either $a = b$ or $a, b \in I$. The same holds for $\mathbb{R}^-$. The characteristic condition is no longer that there are no congruences besides the identity and the total congruence. Rather, it is that every congruence arises has the form $\Theta(I)$.

More generally, given an ℓ-monoid $\mathbf{L}$, a *left (right) ideal* of $\mathbf{L}$ is a non-empty subset I which is both a lattice ideal ($a, b \in I$ implies $a \vee b \in I$) and a left (right) ideal of the monoid reduct: if $i \in I$ and $a \in \mathbf{L}$, then $a \cdot i \in \mathbf{L}$ ($i \cdot a \in \mathbf{L}$). Each left (right) ideal I of $\mathbf{L}$ induces a left (right) congruence of $\mathbf{L}$ as follows:

$$\langle a, b \rangle \in \Theta(I) \iff a \vee i = b \vee i \text{ for some } i \in I.$$

Such congruences will be called *left (right) ideal congruences*. Notice that in an integral tomonoid the left (right) ideals are precisely the downsets, and that each left (right) ideal congruence of a tomonoid has at most one non-trivial congruence class and it is a downset.

An *ideal ℓ-monoid* is an ℓ-monoid where each non-identity left congruence is a left ideal congruence and each non-identity right congruence is a right ideal congruence, excluding the pathological case of ℓ-monoids isomorphic to the two-element additive tomonoid $\{0, +\infty\}$. The following theorem gives a concrete description of ideal tomonoids without a least element. The case of ideal tomonoids with a least element is more complicated to discuss: these are either isomorphic to some $Ł_n$ or to a dense subtomonoid of $[0, 1]$ or – and this is the complicated case – to a certain type of tomonoid over the universe $[0, 1] \cup \{+\infty\}$.

Hölder's theorem for ideal tomonoids without a least element. A tomonoid without a least element is an ideal tomonoid if and only if it is isomorphic to one of the following:

(i) $\mathbb{Z}$,

(ii) $\mathbb{Z}^-$,

(iii) a dense subtomonoid of $\mathbb{R}$,

(iv) a dense subtomonoid of $\mathbb{R}^-$,

In particular, each ideal tomonoid without a least element is commutative.

The restriction to dense subtomonoids which occurs of the above theorem was already implicit in Hölder's theorem for GBL-algebras: every subgroup of $\mathbb{R}$ and more generally every residuated sublattice of $\mathbb{R}$, $\mathbb{R}^-$, and $[0, 1]$ is either dense or isomorphic to $\mathbb{Z}$, $\mathbb{Z}^-$, or $Ł_n$ for some $n \in \mathbb{N}$. In contrast, these algebras have subtomonoids which are neither dense nor isomorphic to $\mathbb{Z}$, $\mathbb{Z}^-$, or $Ł_n$, and which therefore fail to be ideal tomonoids.

To better understand the role of density, consider the subtomonoid $[0, 1/2] \cup \{1\}$ of $[0, 1]$. This is a tomonoid equipped with a drastic multiplication: $1 \odot x = x = x \odot 1$, otherwise $x \odot y = 0$. In this tomonoid, the principal congruence $\langle 1/4, 1/2 \rangle$ is not an ideal congruence, since the equivalence class of $1/2$ is the non-singleton interval $[1/4, 1/2]$, which is not a downset. However, had the tomonoid contained for instance the element $9/10$, its presence would force the equivalence class of $1/2$ to be the interval $[0, 1/2]$ rather than $[1/4, 1/2]$.

The problem of describing all ideal ℓ-monoids, as opposed to merely ideal tomonoids, remains open. We can, however, describe the finite ideal ℓ-monoids.

These coincide with the finite semisimple GMV-algebras, or in other words with finite MV-algebras.

The following result was first formulated as a conjecture by Peter Jipsen. This conjecture is where the condition of being an ideal ℓ-monoid (more precisely, an ideal join-semilattice-ordered commutative monoid) was first isolated.

Theorem. Finite ideal ℓ-monoids are precisely the ℓ-monoid reducts of finite MV-algebras, i.e. up to isomorphism they are the finite products of the ℓ-monoids $Ł_n$ for $n \in \mathbb{N}$.

In particular, all finite ideal ℓ-monoids are reducts of finite GMV-algebras. This does not hold beyond the finite case. However, every ideal ℓ-monoid does satisfy a natural ℓ-monoidal version of the GMV property, namely the last condition in the following equivalence.

Fact. The following are equivalent for every residuated lattice $\mathbf{L}$:

(i) $\mathbf{L}$ is a GMV-algebra, i.e. it satisfies the following equations:

$$x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x.$$

(ii) $\mathbf{L}$ satisfies the following implications for $z \leqslant x \leqslant y$:

$$x\backslash z \leqslant y\backslash z \implies y \leqslant x, \qquad z/x \leqslant z/y \implies y \leqslant x.$$

(iii) $\mathbf{L}$ satisfies the following implications for $z \leqslant x \leqslant y$:

$$(xu \leqslant z \text{ implies } yu \leqslant z) \text{ for all } u \in \mathbf{L} \implies y \leqslant x,$$
$$(ux \leqslant z \text{ implies } uy \leqslant z) \text{ for all } u \in \mathbf{L} \implies y \leqslant x.$$

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The Algebra of Indicative Conditionals

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$\wedge_{\mathsf{OL}}$	**0**	**1/2**	**1**
0	**0**	**0**	**0**
1/2	**0**	**1/2**	**1**
1	**0**	**1**	**1**

$\vee_{\mathsf{OL}}$	**0**	**1/2**	**1**
0	**0**	**0**	**1**
1/2	**0**	**1/2**	**1**
1	**1**	**1**	**1**

	$\neg$
0	**1**
1/2	**1/2**
1	**0**

$\wedge_{\mathsf{K}}$	**0**	**1/2**	**1**
0	**0**	**0**	**0**
1/2	**0**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

$\vee_{\mathsf{K}}$	**0**	**1/2**	**1**
0	**0**	**1/2**	**1**
1/2	**1/2**	**1/2**	**1**
1	**1**	**1**	**1**

$\rightarrow_{\mathsf{OL}}$	**0**	**1/2**	**1**
0	**1/2**	**1/2**	**1/2**
1/2	**0**	**1/2**	**1**
1	**0**	**1/2**	**1**

$\rightarrow_{\mathsf{DF}}$	**0**	**1/2**	**1**
0	**1/2**	**1/2**	**1/2**
1/2	**1/2**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

$\rightarrow_{\mathsf{F}}$	**0**	**1/2**	**1**
0	**1/2**	**1/2**	**1/2**
1/2	**0**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

Figure 1: Tables of the three-valued connectives.

Indicative conditionals are the simplest sentences of the *if-then* type that occur in natural language, concerning what could be true – in opposition to counterfactuals, which concern eventualities that are no longer possible. In Boolean propositional logic, an indicative conditional "if φ then ψ" is traditionally formalized as the material implication $\varphi \to \psi$, or equivalently the disjunction $\neg\varphi \vee \psi$. This approach has several limitations that have been remarked early on in the history of modern logic: in particular, a number of authors argued that conditionals having a false antecedent – which are true in Boolean logic independently of the consequent – should instead be regarded as lacking a (classical) truth value. Such a proposal can be traced back at least to Reichenbach (1935), De Finetti (1936), and Quine (1950). "Uttering a conditional amounts to making a *conditional assertion*: the speaker is committed to the truth of the consequent when the antecedent is true, but committed to neither truth nor falsity of the consequent when the antecedent is false" [1, p. 188]; see also [2] and the references cited therein.

Among various possible ways to formalize the above intuition, a very simple one consists in expanding the classical truth values $(\mathbf{0}, \mathbf{1})$ with a third "gap" value (here denoted by $\mathbf{1/2}$) assigned to conditional sentences with a false antecedent; and then extending the truth tables of the propositional connectives in accordance with the above interpretation. In particular, with regard to the implication, one would certainly require $\mathbf{0} \to x = \mathbf{1/2}$, whereas in other cases (e.g. $\mathbf{1/2} \to x$) intuitions may differ (see Figure 1). As for the designated elements to be preserved in derivations, it is natural to include (besides $\mathbf{1}$) also $\mathbf{1/2}$, at least if one wants to retain basic classical tautologies such as the law of identity $(\varphi \to \varphi)$.[1]

The above constraints determine a range of three-valued propositional *logics of indicative conditionals* which turn out to be, in general, not subclassical (i.e. weaker than) but rather incomparable with classical two-valued logic. In particular, they may be *connexive* in that they validate the (classically contingent) formulas known as *Aristotle's* thesis $\neg(\varphi \to \neg\,\varphi)$ and *Boethius'* theses: $(\varphi \to \psi) \to \neg(\varphi \to \neg\,\psi)$ and $(\varphi \to \neg\,\psi) \to \neg(\varphi \to \psi)$.

Logics of indicative conditionals are discussed at length in the papers [1, 2, 3], which are the main bibliographical source and the starting point for the present research. Here we consider these propositional systems from the standpoint of algebraic logic: in particular, we determine which among them are algebraizable in the sense of Blok and Pigozzi [4], and study the corresponding algebra-based semantics. Besides the ones discussed in [1, 2], we shall also define a few systems obtained by varying the above-mentioned basic parameters (in particular, the designated elements) that do not appear to have been considered in the existing literature; our interest in the latter logics is essentially formal, but future research may prove them to be also relevant to the issues discussed above.

As is well known, a standard way of introducing a propositional logic is to fix an algebra $\mathbf{A}$ together with a subset $D \subseteq A$ of *designated elements* to be preserved in derivations. Such a pair $\langle \mathbf{A}, D\rangle$ is known as a *(logical) matrix*[2], and we may unambiguously denote by $\mathsf{Log}\langle \mathbf{A}, D\rangle$ the propositional consequence relation determined by the matrix $\langle \mathbf{A}, D\rangle$. For the logics of interest here, the universe of the algebra is always going to be the three-element set $A_3 = \{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}$, with variations only in the algebraic operations considered, and possibly the set of designated values. The basic systems are the following (in all cases, we fix $D = \{\mathbf{1/2}, \mathbf{1}\}$):

1. $\mathsf{Log}\langle \mathbf{DF_3}, D\rangle$, where $\mathbf{DF_3} = \langle A_3; \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{DF}}, \neg\rangle$, which is the logic proposed by De Finetti [5]. We show that, up to definitional equivalence, this system coincides with Priest's *logic of paradox* LP [6] expanded with the propositional

[1] A peculiar consequence of this setup is that there will be valid formulas whose negation is also valid: for instance the formula $\neg\,\varphi \to (\varphi \to \varphi)$, which turns out to be equivalent (within the systems considered here) to $\mathbf{1/2}$ viewed as a propositional constant. This makes the logics under consideration not only paraconsistent but actually *contradictory* in the sense of Wansing [13].

[2] See, e.g., [14] for further background on the theory of logical matrices.

constant $\mathbf{1/2}$.

2. $\mathsf{Log}\langle \mathbf{OL_3}, D\rangle$, where $\mathbf{OL_3} = \langle A_3; \wedge_{\mathsf{OL}}, \vee_{\mathsf{OL}}, \to_{\mathsf{OL}}, \neg\rangle$. This is the structural weakening of Cooper's *logic of ordinary discourse* [7], dubbed sOL in the recent papers [8, 9].

3. $\mathsf{Log}\langle \mathbf{CN_3}, D\rangle$, where $\mathbf{CN_3} = \langle A_3; \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{OL}}, \neg\rangle$. A system introduced by Cantwell [10] as the *logic of conditional negation* (CN) and independently considered by a number of other authors[3]. We prove that CN may be viewed as a term-definable subsystem of sOL.

4. $\mathsf{Log}\langle \mathbf{F_3}, D\rangle$, where $\mathbf{F_3} = \langle A_3; \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{F}}, \neg\rangle$, a logic introduced by Farrell [11]. We show that this system is definitionally equivalent to CN (hence, also to a definable subsystem of sOL).

Besides the above systems, we consider a few related ones that, as far as we are aware, have not yet appeared in the literature. These are obtained by:

5. Varying the set D of designated elements on A_3: for instance, logics that result from taking $D = \{\mathbf{1/2}\}$, which is a natural choice at least from a formal standpoint.

6. Considering a set of matrices based on the same algebra. In this way we study *degree-preserving logics* associated to the above-mentioned algebras (see e.g. [12]).

In each case we determine whether the system is algebraizable, thereby settling some issues on the algebraization of logics of indicative conditionals that were raised but left unsolved in [2]. Algebraizable logics are well-behaved in many ways, and in particular one may easily obtain a presentation of the algebraic semantics from an axiomatization of the logic, and vice-versa. In these cases we produce such axiomatizations, and also introduce *twist representations* (akin to that in [9]) that provide further insight into the algebraic semantics; in all the other cases we nevertheless employ algebraic logic techniques to try and obtain some understanding of the models of the logic under consideration.

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[3] As pointed out in [17], this logic – or equivalent systems, with slight variations in the choice of primitive connectives – seems to have been introduced independently in a number of papers from the 1980s to the 2000s (see, e.g., [15, 16]).

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Extending twist construction for Modal Nelson lattices

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This work is about one of the most challenging trends of research in non-classical logic which is the attempt to combine different non-classical approaches together, in our case many-valued and modal logic. This kind of combination offers the skill of dealing with modal notions like belief, knowledge, and obligations, in interaction with other aspects of reasoning that can be best handled using many-valued logics, for instance, vagueness, incompleteness, and uncertainty. In fact, the study that we are going to introduce could be especially interesting from the point of view of Theoretical Computer Science and Artificial Intelligence.
In the present study, we consider the extension of Nelson residuated lattices (N3) with an unary modal operator. We introduce a variety of modal Nelson lattices which we prove that they are characterized by twist structures.

In order to reach this result, we will first introduce an extension for the modal setting of the one well-known construction of Nelson lattices called twist structures, whose importance has been growing in recent years within the study of algebras related to non-classical logics (see [1, 2, 5]). Our proposed extension is more general than others considered in the literature because it is not required to be monotone with respect to modal operators (see [4]).

We assume the reader knows the main properties and definitions about residuated lattices and Heyting algebras. In addition, a residuated lattice is called involutive if it is bounded and it satisfies the double negation equation:

$$a = \neg\neg a.$$

A Nelson residuated lattice or simply Nelson lattice (N3) is an involutive residuated lattice satisfying:

$$((a^2 \to b) \wedge ((\neg b)^2 \to \neg a)) \to (a \to b) = \top.$$

Definition 1. Given a Heyting algebra $\mathbf{H}$, we shall denote by $D(\mathbf{H})$ the filter of dense elements of $\mathbf{H}$, i.e. $D(\mathbf{H}) = \{x \in H : -x = ?\}$.

A filter F of $\mathbf{H}$ is said to be Boolean provided the quotient $\mathbf{H}/F$ is a Boolean algebra. It is well known and easy to check that a filter F of the Heyting algebra $\mathbf{H}$ is Boolean if and only if $D(\mathbf{H}) \subseteq F$. The Boolean filters of $\mathbf{H}$, ordered by inclusion, form a lattice, having the improper filter H as the greatest element and $D(\mathbf{H})$ as the smallest element.
With all these elements, we can reproduce the twist-structures corresponding to N3-lattices.

Theorem 1. *(Sendlewski + Theorem 3.1 in [1].) Given a Heyting algebra*

$$\mathbf{H} = \langle H, \wedge, \vee, \rightharpoonup, \top, ? \rangle$$

and a Boolean filter F of $\mathbf{H}$ let

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = ? \text{ and } x \vee y \in F\}.$$

Then we have:

1. $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \to, ?, \top \rangle$ *is a Nelson lattice, when the operations are defined as follows:*
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \rightharpoonup t) \wedge (s \rightharpoonup y))$,
   - $(x, y) \to (s, t) = ((x \rightharpoonup s) \wedge (t \rightharpoonup y), x \wedge t)$,
   - $\top = (\top, ?)$, $? = (?, \top)$.
2. $\neg(x, y) = (y, x)$,
3. *Given a Nelson lattice $\mathbf{A}$, there is a Heyting algebra $\mathbf{H}_{\mathbf{A}}$, unique up to isomorphisms, and a unique Boolean filter $F_{\mathbf{A}}$ of $\mathbf{H}_A$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H}_{\mathbf{A}}, F_{\mathbf{A}})$.*

Remark 1. Let $\mathbf{A}$ be a Nelson lattice. Let us consider $H = \{a^2 : a \in \mathbf{A}\}$ with the operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in \mathbf{A}$. Then,

$$\mathbf{H}^*_{\mathbf{A}} = \langle H, \vee^*, \wedge^*, \to^*, 0, 1 \rangle$$

is a Heyting algebra ([6]).

Now, for our aim, we need to introduce some definitions of modal algebras.

Definition 2. A modal Heyting algebra M**H** is an algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$ such that the reduct **H** is an Heyting algebra, $\Box$ and $\Diamond$ are two binary operators and, for all $x, y \in H$,

$$\Box x \wedge \Diamond(-x \wedge y) = ? \tag{0.1}$$

Modal Heyting algebras obviously form a variety but it is not very well known. However, there is well known extension of this that is called *normal* modal Heyting algebra. It is obtaned by including the following equations:

3. $-\Diamond x = \Box - x$,
4. $\Box(x \rightharpoonup y) \rightharpoonup (\Box x \rightharpoonup \Box y) = \top$,
5. $\Box \top = \top$.

Note that (1) implies that $\Box x \wedge \Diamond - x = ?$ and $\Box - x \wedge \Diamond x = ?$, therefore, we can conclude $\Diamond - x \leqslant -\Box x$ and $\Box - x \leqslant -\Diamond x$. In addition, if (5) is assumed, we have $\Diamond ? = ?$.

Definition 3. A modal N3-lattice (for short MN3-lattice) is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ such that the reduct **A** is an N3-lattice and, for all $a, b \in A$,

1. $\blacklozenge a = \neg \blacksquare \neg a$,
2. $(\blacksquare a)^2 = (\blacksquare a^2)^2$ and $(\blacklozenge a)^2 = (\blacklozenge a^2)^2$,
3. $(\blacksquare a \wedge \blacklozenge(\neg a^2 \wedge b))^2 = ?$.

In addition, **A** is said to be regular if it satifies the following:

4. $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Moreover, if **A** is a regular modal N3-lattice (for short RMN3-lattice) by using (1) and (4), we can conclude:

4′. $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$.

Finally, we say that a modal Nelson lattice is normal if it is regular and, in addition, satisfies:

5. $\blacksquare \top = \top$.

In this case, we can reproduce the following classical result on RMN3-lattices:

Lemma 1. *If* **A** *is a regular modal N3-lattice then it satisfies the next monotony properties:*

$$\textit{if } a^2 \leqslant b \textit{ then } (\blacksquare a)^2 \leqslant \blacksquare b, \quad \textit{and} \quad \textit{if } (\neg a)^2 \leqslant \neg b \textit{ then } (\neg\blacksquare a)^2 \leqslant \neg\blacksquare b.$$

Now we are ready to formulate the first result of this work.

Theorem 2. *Let* $\mathbf{H}$ *and* F *be a modal Heyting algebra as defined in 2 and a Boolean filter satisfying:*

$$\textit{if } x \wedge y = ? \textit{ and } x \vee y \in F \textit{ then } \Box x \vee \Diamond y \in F.$$

Then, $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \to, ?, \top, \blacksquare, \blacklozenge\rangle$ *is a Modal Nelson lattice, where the operators* $\blacksquare, \blacklozenge$ *are defined as follows:*

$$\blacksquare(x, y) = (\Box x, \Diamond y), \quad \textit{and} \quad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

Now, we are going to extend the representation of Nelson lattices in terms of Heyting algebras from Theorem 1 to the modal context. First, we need to introduce the next result.

Lemma 2. *Let* $\mathbf{A}$ *be a MN3-lattice. Consider* $\mathbf{H}^*_{\mathbf{A}} = \langle H, \vee^*, \wedge^*, \to^*, ?, \top, \Box^*, \Diamond^*\rangle$ *with* $H = \{a^2 : a \in A\}$ *and operators* $\vee^*, \wedge^*, \to^*$ *as in Remark 1 and modal operators as follows*

$$\Box^* a = (\blacksquare a)^2, \quad \textit{and} \quad \Diamond^* a = (\blacklozenge a)^2$$

for every $a \in H$*. Then* $\mathbf{H}^*_{\mathbf{A}}$ *is a modal Heyting algebra. In addition, if we take* $F = \{(a \vee \neg a)^2 : a \in A\}$*, then* F *is a Boolean filter satisfying*

$$\textit{if } a \vee^* b \in F \textit{ and } a \wedge^* b = ? \textit{ then } \Box^* a \vee^* \Diamond^* b \in F$$

for every $a, b \in H$*.*

A direct consequence of previous Lemma is our main result:

Theorem 3. *Let* $\mathbf{A}$ *be a modal N3-lattice. Then* $\mathbf{A}$ *is isomorphic to* $\mathbf{R}(\mathbf{H}^*_{\mathbf{A}}, F)$ *as defined in Theorem 1 by taking* F *as in the previous lemma.*

Now, we would like to finish our presentation by considering two interesting subvariaties of Modal Nelson lattices. First, we consider the modal extension of the subvariety of Nelson lattices, introduced in [3] which is characterized by the following equation:

$$\neg a^2 \to a^2 = (\neg a \to a)^2 \tag{0.2}$$

We denote this modal subvariety by $\mathscr{MNN}$. Let us consider a modal Heyting algebra $\langle \mathbf{H}, \Box, \Diamond\rangle$ such that for all $x \in H$ the following conditions hold:

1. $- - \Box x = -\Diamond - x$,
2. $-\Box - x = - - \Diamond x$.

Then, we are able to prove the following result.

Theorem 4. *A modal Nelson lattice* $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ *satisfies Equation (0.2) if and only if there is a modal Heyting algebra* $\langle \mathbf{H}, \Box, \Diamond \rangle$ *that satisfies conditions* 1. *and* 2. *such that* $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ *is isomorphic to* $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$.

Now, we are going to extend the notion of φ-regular algebra to the modal context. The subvariety of Nelson lattices called φ-regular Nelson lattices were studied in [1] which is characterized by:

$$(\neg a^2)^2 \vee (\neg(\neg a^2)^2)^2 = \top \tag{0.3}$$

We are going to say that $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ is a modal φ-regular Nelson algebra if the non-modal reduct of $\mathbf{A}$ is φ-regular algebra where for any $a \in A$ the unary term $\varphi(a) = (\neg(\neg a)^2)^2 \wedge (\neg(\neg(a \vee \neg a)^2)^2 \vee a)$ satisfies:

$$\blacksquare\varphi(a) = \varphi(\blacksquare a),$$

Definition 4. A modal Heyting algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$ is said to be *crisp-witnessed* if it satisfies the equations

$$- - \Box x = \Box - - x \quad \text{and} \quad - - \Diamond x = \Diamond - - x \tag{0.4}$$

for every $x \in H$.

Thus, we are able to prove the following result.

Theorem 5. *A modal Nelson lattice* $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ *is a modal* φ*-regular Nelson lattice if and only if the associated Heyting algebra* $\langle \mathbf{H_A}, \Box, \Diamond \rangle$ *is crisp-witnessed and satisfies the Stone identity* $-x \vee - - x = \top$.

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Axiomatizing Small Varieties of Periodic ℓ-pregroups

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A *lattice-ordered pregroup* (ℓ-pregroup) is an algebra $(L, \wedge, \vee, \cdot, ^\ell, ^r, 1)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leqslant$ and for every $a \in L$,

$$a^\ell a \leqslant 1 \leqslant aa^\ell \text{ and } aa^r \leqslant 1 \leqslant a^r a.$$

Lattice-ordered pregroups can be seen as a generalization of lattice-ordered groups (ℓ-groups) which have been extensively studied [1, 4, 9]. Indeed, ℓ-groups correspond exactly to the ℓ-pregroups that satisfy $x^\ell \approx x^r$ and in this case x^ℓ is the group inverse operation. On the other hand ℓ-pregroups are a special case of *pregroups* defined similarly to ℓ-pregroups but without demanding that its underlying order is a lattice. Pregroups where introduced in the context of mathematical linguistics [2, 3, 10]. Moreover, ℓ-pregroups are exactly the residuated lattices that satisfy $(xy)^\ell \approx y^\ell x^\ell$ and $x^{r\ell} \approx x \approx x^{\ell r}$, where $x^\ell := x\backslash 1$ and $x^r := 1/x$. So the methods developed for residuated lattices and their connection to substructural logics (see e.g., [8]) also apply to ℓ-pregroups.

An ℓ-pregroup is called *distributive* if its lattice reduct is distributive. The variety DLP of distributive ℓ-pregroups was studied in depth in [5] where a Holland-style representation theorem is obtained and shown that DLP has a decidable equational theory.

In this work we will restrict ourselves to periodic ℓ-pregroups. An ℓ-pregroup is called *n-periodic* for $n \in \mathbb{N}$ if it satisfies the equation $x^{\ell^n} \approx x^{r^n}$. As noted above, 1-preiodic ℓ-pregroups correspond exactly to ℓ-groups. We denote the variety of n-periodic ℓ-pregroups by LP_n. In [7] it was shown that every periodic ℓ-pregroup is distributive. Moreover, in [6] a representation theorem for periodic ℓ-pregroups is obtained and it is shown that the equational theory of LP_n is decidable for each $n \in \mathbb{N}$.

Let $f : \mathbf{P} \to \mathbf{Q}$ and $g : \mathbf{Q} \to \mathbf{P}$ be maps between posets. We say that g is a *residual* for f and f is a dual residual for g if for all $p \in P$, $q \in Q$,

$$f(p) \leqslant q \iff p \leqslant g(q).$$

The residual and dual residual of a map f are unique if they exist and we denote them by f^r and f^ℓ, respectively. Inductively, we define the nth-order residual if it exists, by $f^{r^1} = f^r$ and $f^{r^{n+1}} = (f^{r^n})^r$ and analogously we define the nth-order dual residual of f.

For a chain $\mathbf{\Omega}$ we denote by $F(\mathbf{\Omega})$ the set of maps on $\mathbf{\Omega}$ that have residuals and dual residuals of every order. This set gives rise to a distributive ℓ-pregroup $\mathbf{F}(\mathbf{\Omega}) = (F(\mathbf{\Omega}), \wedge, \vee, \circ, {}^\ell, {}^r, id_\Omega)$, where $\wedge$ and $\vee$ are defined point-wise, $\circ$ is functional composition, and id_Ω is the identity map on Ω. In [5] it was shown that $\mathbf{F}(\mathbb{Z})$ generates DLP. The subset $F_n(\mathbf{\Omega})$ of $\mathbf{F}(\mathbf{\Omega})$ of the maps that satisfy $f^{r^\ell} = f^{r^n}$ forms an n-periodic subalgebra $\mathbf{F}_n(\mathbf{\Omega})$ of $\mathbf{F}(\mathbf{\Omega})$. In contrast to the result about DLP it was shown in [6] that $\mathbf{F}_n(\mathbb{Z})$ does not generate LP_n for any $n \in \mathbb{N}$, but LP_n is generated by $\mathbf{F}_n(\mathbb{Q} \overrightarrow{\times} \mathbb{Z})$ for every $n \in \mathbb{N}$. Nevertheless it was shown that the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is decidable and that $\bigvee_{n\in\mathbb{N}} \mathsf{LP}_n = \bigvee_{n\in\mathbb{N}} \mathbb{V}(\mathbf{F}_n(\mathbb{Z})) = \mathsf{DLP}$, yielding two different ways of approximating DLP with varieties of periodic ℓ-pregroups. A problem left open in [6] is whether the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is finitely axiomatizable. In this work we show that $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized relative to LP_n by a single equation. Let us define $x^{[k]} = x^{\ell^{2k}}$ and $\sigma_n = x \wedge x^{[1]} \wedge \cdots \wedge x^{[n-1]}$. For an n-periodic ℓ-pregroup $\mathbf{L}$ and $a \in L$ the element $\sigma_n(a)$ is exactly the minimal invertible element above a. Our main result can now be stated:

Theorem 1. *For each n the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized relative to LP_n by the equation $x\sigma_n(y)^n \approx \sigma_n(y)^n x$.*

The theorem is proved in three steps. First we connect the congruence lattice of n-periodic ℓ-pregroups to the congruence lattice of their subalgebra of invertible elements which we call the *group skeleton*. In fact the group skeleton is exactly the image of the term operation σ_n. Then, using a decomposition theorem for periodic ℓ-pregroup of [6], we characterize the finitely generated subdirectly irreducible n-periodic ℓ-pregroups that satisfy $x\sigma_n(y)^n \approx \sigma_n(y)^n x$ as lexicographic products of a totally ordered abelian group and $\mathbf{F}_k(\mathbb{Z})$, where k devides n. Finally we show that all of these finitely generated subdirectly irreducibles are contained in $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$. In particular, on the way we obtain the following characterizations of (finitely) subdirectly irreducible members of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$.

Corollary 1. *The finitely generated subdirectly irreducible members of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ are exactly the lexicographic producs of a finitely generated totally ordered abelian group and $\mathbf{F}_k(\mathbb{Z})$ for some k that devides n.*

Corollary 2. *The finitely subdirectly irreducible members of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ are exactly the n-periodic ℓ-pregroups whose group skeleton is a totally ordered abelian group.*

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Fuzzy Logic for Markov Networks

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Introduction

Markov networks, also known as Markov random fields, are a class of probabilistic graphical models that represent the dependencies between random variables using an undirected graph [4, 5]. Markov networks provide a compact representation of high-dimensional probability distributions, coupled with efficient inference algorithms and a clear visual representation. Unlike directed graphical models, such as Bayesian networks, Markov networks are useful for modeling phenomena where directionality cannot be naturally imposed on the relationships between random variables. Markov networks have found widespread application in fields as diverse as image analysis, natural language processing, bioinformatics, and social network analysis.

We argue that certain fuzzy logics related to Hájek's probabilistic logics [3] provide a compact *specification language* for Markov networks. Such a specification language is suitable for formalizing and facilitating reasoning about input from domain experts, which is often essential in practice for building a probabilistic model. In addition, we show that adopting the fuzzy logic perspective clarifies some conceptual issues in the foundations of Markov networks.

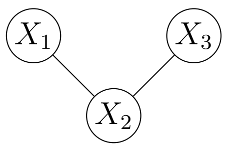
Markov networks

Recall that a **random variable** is a function from a set Ω (the sample space) to a set Val (values). A random variable V is **discrete** if $\mathrm{Val}(V)$ is finite or countably infinite and V is **Boolean** if $\mathrm{Val}(V) = \{0, 1\}$. If $X = (X_1, \ldots, X_n)$ is a tuple of random variables, then $\mathrm{Val}(X)$ is the set of tuples $x = (x_1, \ldots, x_n)$ where $x_i \in \mathrm{Val}(X_i)$.

Definition 1. A **Markov network** is a pair $N = (X, F)$ where $X = (X_1, \ldots, X_n)$ is a tuple of random variables and $F = (\varphi_1, \ldots, \varphi_m)$ is a tuple of **factors** over X, that is, pairs $\varphi_i = (X_{\{i\}}, p_i)$ where $X_{\{i\}}$ is a sub-tuple of X (the scope of φ_i) and $p_i \colon \mathrm{Val}(X_{\{i\}}) \longrightarrow \mathbb{R}^+$ (the potential function). We use the notation $x_{\{i\}}$ for elements of $\mathrm{Val}(X_{\{i\}})$. Tuples $x = (x_1, \ldots, x_n)$ where $x_i \in \mathrm{Val}(X_i)$ are **states** of the network.

A network $N = (X, F)$ can be represented as an undirected graph with vertices X and an edge connecting X_i and X_j iff there is $\varphi \in F$ such that X_i, X_j are in the scope of φ.

Example 5. Let $X = (X_1, X_2, X_3)$ be Boolean and let $F = (\varphi_1, \varphi_2)$ be specified as follows: $X_{\{1\}} = (X_1, X_2)$, $X_{\{2\}} = (X_2, X_3)$ and



X_1	X_2	p_1
1	1	2
1	0	0
0	1, 0	0.5

X_2	X_3	p_2
1	1	0.2
v_2	v_3	1.2
0	0	0.5

where $v_2 + v_3 = 1$.

The factor φ_1 represents an *implication* $X_1 \to X_2$ which prefers states where it is satisfied non-vacuously: states where X_1 is true and X_2 false are ruled out (have value 0), states where X_1 and X_2 are both true have value 2, and states where X_1 is false have lower value 0.5. The factor φ_2 represent a *soft exclusive or* statement where states not satisfying the statement Boolean statement $X_2 \oplus X_3$ are not ruled out, just given lower values. Note that states where both X_2 and X_3 are false are given a slightly lower value (0.2) than states where both are true (0.5).

There is a close relationship between Markov networks and *valued constraint satisfaction problems* [7]. Intuitively, factors correspond to *valued constraints* on states. However, factors play different roles in VCSP and MN, respectively. While VCSP aims at finding the optimal states, the primary role of a Markov network is to provide a compact representation of a probability distribution over states. We focus on networks of finitely discrete random variables.

Definition 2. In $N = (X = (X_1, \ldots, X_n), F = (\varphi_1, \ldots, \varphi_m))$, we define:

Weight of $x \in \mathrm{Val}(X)$	**Normalising constant** of N	**Probability** of $x \in \mathrm{Val}(X)$
$W_N(x) = \prod_{i=1}^{m} \varphi_i(x_{\{i\}})$	$Z_N = \sum_{x \in \mathrm{Val}(X)} W(x)$	$P_N(x) = W_N(x) \cdot Z_N^{-1}$

Probability of events $E \subseteq \mathrm{Val}(X)$ is defined as $P_N(E) = \sum_{x \in E} P_N(x)$ and conditional probability is defined as expected.

Example 6. The weight of state $(1, 1, 0)$ in the network of Example 5 is $\varphi_1(1, 1) \times \varphi_2(1, 0) = 2 \times 1.2 = 2.4$. The normalising constant of the network is 4.35 and the probability of state $(1, 1, 0)$ is approximately 0.55. The probability of the event $X_1 = 1$ is approximately 0.64.

Graded Markov networks

Interpreting the numerical values of factors in Markov networks is a recognised challenge; see, for example, [5, pp. 107–108], [4, p. 106] and [1, p. 269]. Inspired by the connection between Markov networks and valued constraint satisfaction problems, we observe that this problem is mitigated within a particular class of Markov networks.

Definition 3. A Markov network $N = (X, F)$ is **graded** if $\varphi_i(x_{\{i\}}) \in [0, 1]$ for all $\varphi_i \in F$.

We say that two Markov networks $N = (X, F)$ and $N' = (X, F')$ are **equivalent** iff $P(x) = P'(x)$ for all $x \in \mathrm{Val}(X)$.

Theorem 1. *Each discrete Markov network is equivalent to a graded Markov network.*

Proof. It is sufficient to choose a 'big number' B such that $B \geqslant \varphi_i(x_{\{i\}})$ for all $x \in \mathrm{Val}(X)$ and $\varphi_i \in F$, and then define $\varphi_i'(x_{\{i\}}) =_{df} \varphi_i(x_{\{i\}}) \cdot B^{-1}$. □

In graded networks, $\varphi_i(x_{\{i\}})$ can be seen as the result of evaluating, on the arbitrarily fine scale $[0, 1]$, two aspects of the interaction between a state x and a soft constraint φ_i: (i) how well, or to what degree, does x satisfy the constraint φ_i; and (ii) how important is the constraint φ_i? Human users are quite accustomed to scaled assessments (think of film ratings, satisfaction questionnaires, etc.), and so we propose that graded Markov networks provide a format particularly suited to human input.

Example 7. Consider the network in Example 5 and set $B = 2$, the maximal value of a potential function of the network. Then $\varphi_1' = (1, 0, 0.25, 0.25)$ and $\varphi_2' = (0.1, 0.6, 0.6, 0.25)$. The fact that $\varphi_2'(1, 0) = 0.6$ cannot be interpreted to mean that the salient constraint (exclusive or) is satisfied to degree 0.6 in states where X_2 is true and X_3 is false. The constraint is fully satisfied, it's just that it's considered less important than φ_1'. If necessary, we can explicitly state the *importance degrees* of the constraints, e.g. 1 for φ_1' and 0.6 for φ_2'.

In addition, using $[0, 1]$ instead of $\mathbb{R}^+$ (or even $\mathbb{R}^{+\infty}$) enables a better grasp on the relative sizes of degrees. Indeed, while all closed intervals of real numbers are homeomorphic, in $[0, 1]$ we have a fixed standard arithmetics that allows e.g. to see the number 0.5 as the middle point of the interval in a canonical way, whereas the exact location of the middle point of $\mathbb{R}^{+\infty}$ would actually depend on what particular homeomorphism one uses to map it back to $[0, 1]$.

Symbolic representation I

In what follows, we focus on **graded Boolean** networks (GB), that is, graded Markov networks where all random variables are Boolean. GB networks have a straightforward *symbolic representation* in terms of **weighted Boolean theories**, that is, tuples of pairs $((\alpha_1, w_1), \ldots, (\alpha_r, w_r))$, where α_i is a Boolean formula and $w_i \in [0,1]$ for all $i = 1, \ldots, r$. Starting with a GB network $N = (X, F)$, each factor $\varphi_i \in F$ over Boolean variables X_j for $j \in J_i$ corresponds to $T_i = ((\alpha_j, w_j)$ for $j \in J_i)$ where each α_j is a Boolean formula over X representing one value assignment to Boolean variables in the scope of φ_i and w_j is the value of φ_i under that assignment. The weighted theory T_N is just a concatenation of the tuples T_i for $\varphi_i \in F$.

Example 8. The network of Example 7 corresponds to the concatenation of

$$\begin{gathered}\big((X_1 \wedge X_2,\, 1), \quad (X_1 \wedge \neg X_2,\, 0),\\ (\neg X_1 \wedge X_2,\, 0.25), \quad (\neg X_1 \wedge \neg X_2,\, 0.25)\big)\end{gathered} \quad \text{and}$$
$$\begin{gathered}\big((X_2 \wedge X_3,\, 0.1), \quad (X_2 \wedge \neg X_3,\, 0.6),\\ (\neg X_2 \wedge X_3,\, 0.6), \quad (\neg X_2 \wedge \neg X_3,\, 0.25)\big)\end{gathered}$$

The weighted theory can be simplified by leaving out formulas with weight 1 and forming disjunctions of mutually inconsistent formulas with the same weight:

$$\big((X_1 \wedge \neg X_2,\, 0), (\neg X_1,\, 0.25), (X_2 \wedge X_3,\, 0.1), (X_2 \leftrightarrow \neg X_3,\, 0.6), (\neg X_2 \wedge \neg X_3,\, 0.25)\big).$$

Conversely, every weighted theory $T = ((\alpha_j, w_j)$ for $j \in J)$ can be converted into a GB network $N_T = (X_T, F_T)$ where X_T is the tuple of all propositional variables appearing in α_j for $j \in J$ and $F = (\varphi_j$ for $j \in J)$ such that the scope of φ_j are the variables appearing in α_j and $\varphi_j(x_{\{j\}}) = w_j$ if α_j is satisfied in x and $= 1$ otherwise, for all $j \in J$. Hence, every formula is considered as a separate factor. The symbolic representation of Markov networks in terms of weighted theories is the key idea behind *Markov Logic Networks* [2, 6]. In this case, however, the formulas are of first-order logic, not propositional logic. The idea, however, is quite general and can be applied to various other languages, for example variants of modal logics.

Symbolic representation II

A similar symbolic representation can be set up using a variant of the *two-layered fuzzy logic of probability* by Hájek et al. [3]. The set of formulas Fm is defined by

$$\varphi, \psi := X_i \mid \bar{0} \mid \varphi \to \psi \mid \varphi \odot \psi \mid \mathsf{F}(\alpha) \mid \mathsf{P}_T(\alpha)\,,$$

where α does not contain F nor P (it is a Boolean formula over the set of variables $Pr = \{X_i \mid i \in \mathbb{N}\}$) and T is a finite set of Boolean formulas. Other connectives are defined as usual. A model is $M = (S, f, V)$, where S is a non-empty countable set, $V\colon Pr \longrightarrow 2^S$ and $f\colon 2^S \longrightarrow [0,1]$. An M-interpretation is a function $[\![\cdot]\!]\colon S \times Fm \longrightarrow [0,1]$ such that (let $\|\varphi\| = \{s \mid [\![\varphi]\!]_s = 1\}$)

$$[\![X_i]\!]_s = \begin{cases} 1 & \text{if } s \in V(X_i) \\ 0 & \text{otherwise} \end{cases} \qquad [\![\bar{0}]\!]_s = 0 \qquad [\![\varphi \odot \psi]\!]_s = [\![\varphi]\!]_s \cdot [\![\psi]\!]_s$$

$$[\![\varphi \to \psi]\!]_s = \min\{1, 1 - [\![\varphi]\!]_s + [\![\psi]\!]_s\} \qquad [\![\mathsf{F}(\alpha)]\!]_s = f\|\alpha\|$$

$$[\![\mathsf{P}_T(\alpha)]\!]_s = \textstyle\sum_{s \in \|\alpha\|} \mathsf{Prob}(s),$$

($\mathsf{Prob}_T(s) = \mathsf{Weight}_T(s) \cdot Z_T^{-1}$, $\mathsf{Weight}_T(s) = \prod_{\alpha \in T} [\![\alpha \to \mathsf{F}(\alpha)]\!]_s$ and $Z_T = \sum_{s \in S} \mathsf{Weight}_T(s)$).

A finite set T of Boolean formulas is a **template**, expressing the structure of a Boolean graded network. Given a model and a template T, the tuple $((\alpha, f\|\alpha\|)$ for $\alpha \in T)$ gives rise to a weighted-theory representation of a specific network with template T. Formulas $\mathsf{F}(\alpha)$ represent **parameters** of the network with values $[\![\mathsf{F}(\alpha)]\!]$ (note that these do not depend on $s \in S$).

In practice, parameter values can often be provided by domain experts or learned from data. Often, a combination of these two approaches is the most efficient strategy to obtain a network that approximates a probability distribution: a domain expert provides information about the network without fully specifying it, and the rest is learned from data, with the hypothesis space constrained by the expert input. In our view, fuzzy logic provides a compact *specification language* for such expert input.

Example 9. Formula $\mathsf{F}(\alpha_1) \to \mathsf{F}(\alpha_2)$ says that (has truth degree 1 iff) the value of parameter $\mathsf{F}(\alpha_1)$ is less or equal than the value of the parameter $\mathsf{F}(\alpha_2)$. If we extend the language with constants $\bar{c}$ for $c \in [0,1] \cap \mathbb{Q}$, then we can express that a given parameter is less (greater) or equal to c by formulas of the form $\mathsf{F}(\alpha) \to \bar{c}$ (resp. $\bar{c} \to \mathsf{F}(\alpha)$). Similarly, $\mathsf{P}_T(\alpha) \to \mathsf{P}_{T'}(\alpha)$ means that the probability of α in any network with structure T is less or equal to the probability of α in any network with structure T'.

Finding a complete axiomatization and determining decidability of our logic are natural open problems.

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Relative ideals, homological categories and non-classical logics

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Basic terminology in homological category theory

We briefly recall the following key concepts. We refer the interested reader to [2, 3] for a comprehensive account.

A category is *pointed* if it has a zero-object, that is, an object 0 that is both terminal and initial. In a pointed category, a *kernel* of a map $p\colon A \to B$ is the pullback of the initial map $0 \to B$ along p.

Kerar$[r]$ar$[d]$ Aar$[d]^p$

0ar$[r]$ B

More generally, a *kernel pair* of p is the pullback of p with itself. An (internal) equivalence relation is called *effective* when it is the kernel pair of a morphism. A category B is *regular*, if it is finitely complete, has pullback-stable regular epimorphisms, and all effective equivalence relations admit coequalizers. A regular category is *Barr-exact* [1] when all equivalence relations are effective.

We will also need the concept of *protomodularity* whose technical definition goes beyond the scope of this abstract. Intuitively, a category is called *protomodular* if it possesses an intrinsic notion of *normal subobject* —in analogy with normal subgroups. The notion of protomodular (and semi-abelian) category encompasses a wide range of categories of interest to algebraists, including categories of groups, rings, Lie algebras, associative algebras, and cocommutative Hopf algebras, among others. Of course, all abelian categories are semi-abelian. We refer the reader to [4] for the original definition and to [5] for a more general account.

A category is called *homological* if it is pointed, regular and protomodular. Many of the standard results of classical homological algebra hold in homological categories.

Finally, a category is *semi-abelian* if it is homological, Barr-exact, and has finite coproducts.

Introduction

In the algebraic semantics of non-classical logics, often there are two constants in the language, one for absolute truth and one for absolute falsehood. As a consequence, the final object (=the trivial algebra) is different from the initial one (=the ∅-generated free algebra). This makes the categories at play non-pointed and thus apparently intractable with homological methods. We report on the article [9], which aims at showing that one may still connect the equivalent algebraic semantics of non-classical logics to homological categories. The idea is best explained by the following example from classical algebra.

The forgetful functor

$$U\colon \mathsf{CRing} \to \mathsf{CRng}\,,$$

from the protomodular category of commutative unital rings to the semi-abelian category of commutative rings has a left adjoint F that freely adds the multiplicative identity. In greater detail, if R is a commutative ring, then $F(R)$ has underlying abelian group given by the direct product $R \times \mathbb{Z}$, endowed with a multiplication defined by the formula

$$(r,n)(r',n') := (rn' + nr' + rr', nn') \tag{0.1}$$

and with multiplicative identity $(0,1)$. It is easy to see that the ring R is contained in $F(R)$ as an ideal. By the arbitrary choice of R, this means that every commutative ring can be seen as an ideal of a suitable *unital* commutative ring. More precisely, one can prove that there is an equivalence of categories $\mathsf{CRing}/\mathbb{Z} \simeq \mathsf{CRng}$ which is defined by taking kernels in CRng of the "objects" in the slice category.

Now, it is well known that ideals of commutative unital rings make it easier to deal with congruences (and quotients) in CRing, but since they are not subobjects in CRing, it is not immediate to describe them categorically. On the other hand, ideals of commutative unital rings are subobjects in CRng. This makes it possible to exploit the categorical properties of the semi-abelian category CRng in order to study the, still protomodular but not pointed, category CRing.

U-ideals and the basic setting

We propose to set a study of these facts in a more general framework. As a first step, we introduce the notion of U*-ideal*. Let A be a pointed category with pullbacks and $U\colon \mathsf{B} \to \mathsf{A}$ be a faithful functor. Intuitively, a U-ideal is a kernel in A of a map that lives in B. A formal definition follows.

Definition 1. Let B be an object in B. A morphism $k\colon A \to U(B)$ in A is called *U-ideal* of B if there exists a morphism $f\colon B \to B'$ of B that makes the following diagram a pullback in A:

$$A\mathsf{ar}[r]^{-}k\mathsf{ar}[d] \qquad U(B)\mathsf{ar}[d]^{U(f)}$$

$$0\mathsf{ar}[r] \qquad U(B')$$

Since kernels are monomorphisms, U-ideals can be seen as subobjects in a "larger" category. Prototypical examples of U-ideals come from the inclusion $U\colon \mathsf{Ring} \to \mathsf{Rng}$ of unital rings into rings: U-ideals in Ring are just the usual bilateral ideals of ring theory.

The example from ring theory suggests considering a more robust environment for dealing with the notion of U-ideal.

Definition 2. A *basic setting* for relative U-ideals is an adjunction

$$\mathsf{Bar}[r]\_U \qquad \mathsf{Aar}@/_3ex/[l]^{?}_{F}\,, \tag{0.2}$$

where the category A is homological and U is a conservative faithful functor.

We develop these ideas below. Our main results are the following. An equivalence, in the varietal settings, between relative U-ideals and Ursini's 0-ideals [7] (see Theorem 2). In Section we describe a more general equivalence (Theorem 3) and then we apply it to the case of MV-algebras, showing an equivalence between Weisberg hoops and filters of MV-algebras (see Corollary 1). In order to apply our framework to MV-algebras we show that the category of Hoops is homological (see Theorem 4), thus showing that our setting applies to a number of non-classical logics.

U-ideals and varieties of algebras

Let us recall the characterization of protomodular varieties, as established by Bourn and Janelidze in [6].

Theorem 1. *A variety V of universal algebras is protomodular if and only if there is a natural number n, 0-ary terms $e_1, \ldots, e_n$, binary terms $\alpha_1, \ldots, \alpha_n$, and $(n+1)$-ary term θ such that:*

$$\begin{aligned} &\mathsf{V} \models \theta(\alpha_1(x,y), \ldots, \alpha_n(x,y), y) = x && \textit{and} \\ &\mathsf{V} \models \alpha_i(x,x) = e_i && \textit{for } i = 1, \ldots, n\,. \end{aligned} \tag{0.3}$$

Let us also recall the following definitions introduced by Ursini (see [7] and references therein).

Definition 3. Let V be a variety with a constant symbol 0 in its signature Σ_{V}. A Σ_V-term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is called 0-*ideal term* in the variables $y_1, \ldots, y_n$ if

$$\mathsf{V} \models t(x_1, \ldots, x_m, 0, \ldots, 0) = 0\,.$$

For any algebra A in V, a subset $\varnothing \neq H \subseteq A$ is called 0-*ideal* if for every $a_1, \ldots, a_m \in A$, any $h_1, \ldots, h_n \in H$, and every 0-ideal term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in the variables $y_1, \ldots, y_n$, one has $t(a_1, \ldots, a_m, h_1, \ldots, h_n) \in H$.

It turns out that in any algebra in V the equivalence class of 0 under any given congruence is a 0-ideal. Vice versa, one calls 0-*ideal determined*, or just *ideal determined* (cf. [7, Definition 1.3]) a variety V where every 0-ideal is the equivalence class of 0 for a unique congruence relation.

A relevant subclass of the class of ideal determined varieties is the class of the so-called *classically ideal determined*. They are the varieties satisfying the conditions of Theorem 1, with all e_i's equal to a single constant $0 \in \Sigma_{\mathsf{V}}$.

In order to compare our notion of U-ideal and Ursini's notion of 0-ideal, we specialize the basic setting of Definition 2 to the varietal case.

Definition 4. A *basic setting for varieties* is given by a functor $U\colon \mathsf{B} \to \mathsf{A}$ between two varieties of algebras such that: 1. the axioms of B extend the axioms of A, possibly in a larger language; 2. the category A is homological (thus semi-abelian); 3. the functor U is the obvious forgetful functor.

Notice that, according to [2, Proposition 3.5.7 and Theorem 3.7.7], U is a faithful conservative right adjoint, thus satisfies the conditions of Definition 2. Obviously, if $U\colon \mathsf{B} \to \mathsf{A}$ is a basic setting for varieties, then A is classically ideal determined. The next result establishes a connection between the varietal notion of 0-ideal and the categorical notion of U-ideal.

Theorem 2. *Let $U\colon \mathsf{B} \to \mathsf{A}$ be a basic setting for varieties. A subset H of an algebra B of B is a 0-ideal of B if and only if $H \subseteq U(B)$ is a U-ideal of B with respect to $U\colon \mathsf{B} \to \mathsf{A}$.*

Categorical equivalence and non classical logics

Let us go back to the general situation and consider a basic setting as in (0.2). For every object A of A, the unit of the adjunction η gives a universal morphism $\eta_A\colon A \to UF(A)$. Thus, for any B in B the unique morphism $0\colon A \to 0 \to U(B)$ factors through η_A as in the diagram below:

$$F(A)\mathsf{ar}[d]_{\exists! p_A}^{\text{such that}} \qquad A\mathsf{ar}[r]^{-}\eta_A\mathsf{ar}[dr]\_0 \qquad UF(A)\mathsf{ar}@-- >[d]^{U(p_A)} \tag{0.4}$$

$$B \qquad\qquad\qquad U(B)$$

Definition 5. We say that η_A is an *augmentation U-ideal*, or more simply an *augmentation ideal*, if it is the kernel of $U(p_A)$.

Theorem 3. *Suppose that in the basic setting (0.2) for every A in* A*, the component η_A is an augmentation ideal. Then, the kernel functor*

$$K \colon \mathsf{B}/B \to \mathsf{A}$$

defined on objects by letting $K(f) := Ker(U(f))$, is an equivalence of categories.

Finally, we apply Theorem 3 to the setting of MV-algebras. First, generalising a proof in [8] we provide terms as requested by Theorem 1, thus obtaining the following result.

Theorem 4. *The variety* Hoops *is semi-abelian.*

Consequently, our setting applies to certain categories of interest in algebraic logic, such as Wajsberg, Product, and Gödel hoops, with respect to MV, Product and Gödel algebras. In particular, an application of Theorem 3 gives the following.

Corollary 1. *The kernel functor $K \colon \mathsf{MVAlg}/\mathbf{2} \to \mathsf{WHoops}$ defined by sending an arbitrary homomorphism $f \colon A \to \mathbf{2}$ into its kernel $Ker(f)$, is an equivalence of categories.*

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The structure and theory of McCarthy algebras

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In his seminal paper [5], in regards to the theory of computation, John McCarthy introduced a logic for computable functions with the aim of managing *undefined* assignments, partial predicates, and modeling computational failures. As the order in which programs are executed may be paramount, the conjunction/disjunction with an undefined value may fail to commute, and thus yields a *non-commutative* logic. This paradigm has also found application in the study of Process Algebras, such as the handling and management of *errors* in concurrent programming; for instance in [1] where the operation $\cdot$ in Figure 1 is used for *left sequential conjunction.*

The first algebraic treatment for a 3-valued semantics of McCarthy's logic was carried out by Konikowska in [4], where the following operation tables over a set $M_3 := \{0, 1, \varepsilon\}$ are introduced.

$'$	
1	0
0	1
ε	ε

$+$	1	0	ε
1	1	1	1
0	1	0	ε
ε	ε	ε	ε

$\cdot$	1	0	ε
1	1	0	ε
0	0	0	0
ε	ε	ε	ε

Figure 1: The operation tables for the algebra $\mathbf{M}_3 := \langle \{0, 1, \varepsilon\}, +, \cdot, ', 0, 1\rangle$.

As Konikowska defines in [4], an algebra $\langle A, +, \cdot, ', 0, 1\rangle$ is called a **McCarthy algebra** if it *"satisfies all the equational tautologies of a Boolean algebra that hold in"* the algebra $\mathbf{M}_3$. From the observation that the two-element Boolean algebra $\mathbf{2}$ is a subalgebra of $\mathbf{M}_3$, we may restate this, within the parlance of universal algebra, and understand a McCarthy algebra to be any member in the *variety* of algebras generated by $\mathbf{M}_3$. In this way, let us define M to be the **variety of McCarthy algebras** denoting $\mathsf{V}(\mathbf{M}_3)$.

The following properties are readily verified for the algebra $\mathbf{M}_3$, and thus also M:

- the operation $'$ is an *involution*, i.e., $x'' \approx x$, through which the constants $0 \approx 1'$ and $1 \approx 0'$ are inter-definable;
- the operations $+$ and $\cdot$ the term-definable from each other through $'$ via $x + y \approx (x' \cdot y')'$ and $x \cdot y \approx (x' + y')'$, i.e., they satisfy the De Morgan laws;
- the reduct $\langle M_3, \cdot, 1\rangle$ (thus also $\langle M_3, +, 0\rangle$) is a monoid with an *idempotent* operation, i.e., $x \cdot x \approx x$ (thus also $x + x \approx x$).

Let us call an algebra $\langle A, \cdot, ', 1\rangle$ an *unital band with involution* (**i-uband** for short) if $\langle A, \cdot, 1\rangle$ is a unital band (i.e., idempotent monoid) and $'$ an involution on A; we write $\langle A, +, \cdot, ', 0, 1\rangle$ to indicate its term-definable De Morgan dual $\langle A, +, ', 0\rangle$ in the signature.

Theorem 1. *There are exactly ten non-isomorphic i-ubands of cardinality 3, exactly four of which containing* **2** *as a Boolean subalgebra; the Strong Kleene algebra* **SK**, *the Weak Kleene algebra* **WK**, *the McCarthy algebra* $\mathbf{M}_3$ *and its mirror* $\mathbf{M}_3^{\mathsf{op}}$ *(i.e., where* $x \cdot^{\mathsf{op}} y := y \cdot x$*).*

While a great deal is known about the Strong and Weak Kleene algebras and the varieties they generate (see e.g. [6, 3, 1]), little is known about the variety M of McCarthy algebras. In the same article [4], Konikowska gives a long list of equational identities that are valid for M, but whether this list forms a complete axiomatization is left open as conjecture. Part of this research settles this question by both demonstrating that Konikowska's identities are indeed complete for M, and also providing a number of equivalent and minimal axiomatizations. We motivate one such presentation as follows.

For one, the algebra $\mathbf{M}_3$ satisfies distributivity from the left:

$$x \cdot (y + x) \approx xy + xz \quad \text{(or, equivalently)} \quad x + yz \approx (x + y) \cdot (x + z) \qquad \text{(left-distributivity)}$$

However, $\langle M_3, +, \cdot\rangle$ is not a semiring as distributivity from the right fails in general. But some instances of this law do hold, in particular the following:

$$(x + x') \cdot y \approx xy + x'y \quad \text{(or, equivalently)} \quad xx' + y \approx (x + y) \cdot (x' + y) \qquad \text{(ortho-distributivity)}$$

Of course, the most glaring identity that fails in $\mathbf{M}_3$ is that of commutativity. Thus the monoid reduct fails to form a semi-lattice. Even worse, $\langle M_3, +, \cdot\rangle$ is not even a skew-lattice, as the right-absorption laws are falsified (e.g., $1 \neq (\varepsilon + 1) \cdot 1 = \varepsilon$). However, $\mathbf{M}_3$ does satisfy the following left-absorption law:

$$x \cdot (x + y) \approx x \quad \text{(or, equivalently)} \quad x + xy \approx x \qquad \text{(left-absorption)}$$

While $\mathbf{M}_3$ is not ortho-complemented, i.e., the identity $1 \approx x + x'$ (equivalently, $0 \approx x \cdot x'$) fails, it does satisfy a *local* version with unary term-operations $0_x := x \cdot 0$ and $1_x := x + 1$:

$$1_x \approx x + x' \quad \text{(or, equivalently)} \quad 0_x \approx x \cdot x' \qquad \text{(locally complemented)}$$

Lastly, while commutativity generally fails, it does satisfy some instances. In particular for the *local units* $1_x := x + 1$ and $0_x := x \cdot 0$:

$$1_x \cdot 1_y \approx 1_x \cdot 1_y \quad \text{(or, equivalently)} \quad 0_x + 0_y \approx 0_x + 0_y \quad \text{(local-unit commutativity)}$$

Definition 1. We call a *McCarthy-Konikowska algebra* (**MK-algebra**) any i-uband satisfying left-distributivity, ortho-distributivity, left-absorption, locally complemented, and local-unit commutativity. Denote the variety of MK-algebras by MK.

With a good deal of work, we verify the following:

Theorem 2. *Konikowska's axioms [4, (A1–A16) pp. 169] hold in* MK.

Among these identities sits that of *left-regularity*, i.e., $xyx \approx xy$. In fact, and while the derivation is far from trivial, any left-distributive i-uband satisfying local-unit commutativity is also left-regular. As is well-known, any left-regular operation $*$ admits a partial order $\leqslant_*$ defined via $x \leqslant_* y$ iff $x * y = y$. For MK-algebras, we choose to work with the partial order associated with the operation $+$, and will denote it simply by $\leqslant$. This fact affords us the following structure theorem for MK-algebras. First, recall the standard notation $\uparrow a := \{x \in A : a \leqslant x\}$ and $\downarrow b := \{x \in A : x \leqslant b\}$, and that of an interval $[a, b] := \uparrow a \cap \downarrow b$.

Theorem 3. *Let* $\mathbf{A} = \langle A, +, \cdot, ', 0, 1\rangle$ *be an MK-algebra. Define* $\mathscr{I}_{\mathbf{A}} := \{0_a : a \in A\}$ *and, for each* $i \in \mathscr{I}_{\mathbf{A}}$*, set* $B_i := [0_i, 1_i]$*, where* $0_x := x \cdot 0$ *and* $1_x := x + 1$*. Then the following hold:*

1. $\langle \mathscr{I}_{\mathbf{A}}, \vee, 0\rangle$ *is a join-semilattice with least element* 0*, where* $i \vee j := i + j$*.*
2. *For each* $i \in \mathscr{I}_{\mathbf{A}}$*,* $\mathbf{A}_i := \langle \uparrow 0_i, +, \cdot, ', 0_i, 1_i\rangle$ *is an MK-algebra and the map* $h_i : x \mapsto 0_i + x$ *is a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}_i$*.*
3. *For each* $i \in \mathscr{I}_{\mathbf{A}}$*, the structure* $\mathbf{B}_i := \langle B_i, +, \cdot, ', 0_i, 1_i\rangle$ *is a Boolean algebra and the set* B_i *coincides with* $\{x \in A : 0_x = 0_i\}$*. Consequently,* $A = \bigcup_{i \in \mathscr{I}_{\mathbf{A}}} B_i$ *and the members of* $\{B_i\}_{i \in \mathscr{I}_{\mathbf{A}}}$ *are pairwise disjoint.*
4. *For each* $i, j \in \mathscr{I}_{\mathbf{A}}$ *with* $i \leqslant j$*, the map* $\rho_{ij} := h_i \restriction B_i$ *is a homomorphism from* $\mathbf{B}_i$ *to* $\mathbf{B}_j$*. Moreover,* $\rho_{ii} = \mathrm{id}_{\mathbf{B}_i}$ *and* $\rho_{jk} \circ \rho_{ik} = \rho_{ik}$ *for each* $i \leqslant j \leqslant k$ *in* $\mathscr{I}_{\mathbf{A}}$*.*

This structure theorem allows for a finer analysis of MK-algebras, in particular those that are subdirectly irreducible, and ultimately serves as the linchpin for the following characterization.

Theorem 4. *The only subdirectly irreducible MK-algebras are the two-element Boolean algebra* $\mathbf{2}$ *and the* 3*-element MK-algebra* $\mathbf{M}_3$*.*

As every variety of algebras is generated by its subdirectly irreducible members, and $\mathbf{2}$ is a subalgebra of $\mathbf{M}_3$, we immediately obtain the following as a corollary to Theorem 4.

Corollary 1. *The variety of MK-algebras is generated by the algebra* $\mathbf{M}_3$*. Consequently, the variety of McCarthy algebras coincides with* MK*.*

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The cardinality of intervals of modal and superintuitionistic logics

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Introduction

We study the cardinalities of intervals of modal and superintuitionistic logics (si-logics for short). This cardinality cannot be more than the continuum as we assume that our language is countable, and in a countable language we cannot have more than a continuum of logics (which are special sets of formulas). Recall that for modal or si-logics L_1 and L_2 the interval $[L_1, L_2]$ is the set

$$[L_1, L_2] = \{L : L_1 \subseteq L \subseteq L_2\}.$$

These intervals are, clearly, not linearly ordered.

It was first shown by Jankov [6] that there are continuum many si-logics. Therefore, the intervals [IPC, $\mathsf{Inconsist}$] and [IPC, CPC], where IPC and CPC are intuitionistic and classical propositional calculi, respectfully, and $\mathsf{Inconsist}$ is the inconsistent logic, have the cardinalities that of the continuum. This was obtained by constructing an antichain of finite subdirectly irreducible Heyitng algebras (alternatively, finite rooted posets) with respect to homomorphic image of a subalgebra order (alternatively, p-morphic image of an upset) and by associating to each such finite algebra the so-called *Jankov formula*, a variant of the diagram of this algebra [6], see also [1]. These results have been generalized to modal logics by Fine [4] and Rautenberg [8] (see [3, Chapter 9] for an overview). For example, the intervals [$\mathsf{S4}$, $\mathsf{Inconsist}$] and [$\mathsf{K4}$, $\mathsf{S4}$] have the cardinality that of the continuum. It is also known that there are some intervals that are finite, e.g., extensions of any tabular transitive modal or si-logic and countably infinite, e.g, the intervals [$\mathsf{S4.3}$, $\mathsf{Inconsist}$] or [LC, $\mathsf{Inconsist}$] (see e.g., [3]).

It was posed as an open problem, only very recently in [5], whether it can be proved without assuming the Continuum Hypothesis (CH) that each interval of modal logics has the cardinality which is countable or that of the continuum. In particular, suppose that for modal or si-logics L_1 and L_2, the interval $[L_1, L_2]$ is not countable, then is the cardinality of this interval that of the continuum, without the use of the CH? This question was triggered by investigations into the degrees of the finite model property (the FMP). This concept was defined in [5] and it was shown that the degree of the FMP of each transitive modal or si-logic can be any finite cardinal, $\aleph_0$ or $2^{\aleph_0}$. With the CH this implies that any cardinality $\leqslant 2^{\aleph_0}$ can be the degree of the FMP for some transitive modal or si-logic. Because of this, this result was called the Antidichotomy theorem. It was also shown in [5] that the degrees of the FMP for these logics always form an interval.

In this paper, we resolve this open problem affirmatively. We prove this by using techniques from descriptive set theory (see e.g., [7, Section 12]). Specifically, we represent the set of propositional variables as natural numbers and logics as reals. Then sets of logics correspond to some sets of real numbers. We show that for any interval of modal or si-logics the corresponding set of reals is $\mathbf{\Pi}^0_1$, in particular, it is a Borel set. It is a classical result in descriptive set theory that every Borel set has the perfect set property [7]. Thus, the cardinality of such a set is either countable or continuum. As a result, we obtain that every uncountable interval of logics has the cardinality that of the continuum, and the degree of FMP of any transitive modal logic or si-logic can be only any finite cardinal, $\aleph_0$ or $2^{\aleph_0}$ without assuming the CH. This gives the solution to our problem. We also provide a direct proof showing that the degree of FMP in the lattice of normal extensions of any normal modal logic is $\mathbf{\Pi}^0_2$, so the cardinality result also holds for non-transitive modal logics.

As far as we are aware, this perspective on the study of intervals of logics has not been explored before.

Main results and proof sketches

We will now move to formal details.

Definition 1. Let L_0 be a normal modal logic.

1. Let $\mathsf{NExt}L_0$ be the lattice consisting of all normal extensions of L_0, namely all normal logics containing L_0, with the order $\subseteq$.
2. Given $L_1 \in \mathsf{NExt}L_0$, let
$$[L_0, L_1] = \{L \text{ normal logic} : L_0 \subseteq L \subseteq L_1\} = \{L \in \mathsf{NExt}L_0 : L \subseteq L_1\}.$$
3. Let FFr be the set of all finite Kripke frames. For $L \in \mathsf{NExt}L_0$, let $\mathsf{FFr}(L) = \{F \in \mathsf{FFr} : F \vDash L\}$, and $\mathsf{fmp}_{L_0}(L) = \{L' \in \mathsf{NExt}L_0 : \mathsf{FFr}(L') = \mathsf{FFr}(L)\}$.
4. The degree of fmp of L in $\mathsf{NExt}L_0$ is the cardinality of the set $\mathsf{fmp}_{L_0}(L)$.

Remark 1. These definitions also apply to si-logics.

Our central observation is that we can identify formulas with natural numbers, logics (which are sets of formulas) with real numbers, and intervals (which are sets of logics) with sets of reals. Since there are countably many proportional variables, we can encode modal formulas in an effective way, that is, $\mathrm{Fml} = \{i \in \omega : i \text{ is a code of a modal formula}\}$ is recursive. φ_i will denote the formula with code i. Similarly, given some $A \subseteq \mathrm{Fml}$, let $L_A = \{\varphi_i : i \in A\}$. Note that every formula has a unique code and every logic has a unique set of codes. Explicitly, we have $i \in \mathrm{Fml}$ and $\varphi_i \in L$ iff $i \in A$, for all $i \in \omega$. We further identify subsets of ω with reals, namely, elements in the Cantor space 2^ω in the canonical way. Under this identification, in particular, logics correspond to elements of 2^ω, and sets of logics to subsets of 2^ω.

This allows us to investigate the arithmetical and Borel complexity of sets of logics, viewed as sets of reals, and apply facts from descriptive set theory (see e.g., [7, Section 12]).

Definition 2. Let $f, g \in 2^\omega$. $f \oplus g \in 2^\omega$ is defined by

$$(f \oplus g)(2n) = f(n) \text{ and } (f \oplus g)(2n+1) = g(n), \text{ for all } n \in \omega.$$

Lemma 1. *Let $L_1 \in \mathsf{NExt}L_0$ and $A_1 \subseteq Fml$ be the code of L_1. Let $[A_0, A_1] = \{A \subseteq Fml : L_A \in [L_0, L_1]\}$. Then $[A_0, A_1] \in \Pi^0_1(A_0 \oplus A_1)$. Moreover, if L_0 is recursively axiomatizable and L_1 is decidable, then $[A_0, A_1] \in \Pi^0_1$.*

Proof Idea. The set $[A_0, A_1] \subseteq 2^\omega$ can be defined by a Π^0_1 formula with parameters A_0, A_1. For example, for $A \subseteq \mathrm{Fml}$, being closed under necessitation is characterized by $\forall i \forall j(\mathrm{Nec}(i, j) \wedge j \in A \to i \in A)$, where $\mathrm{Nec}(x, y)$ is a recursive relation such that $\mathrm{Nec}(i, j)$ iff $i, j \in \mathrm{Fml}$ and φ_i is of the form $\Box\varphi_j$; being an extension of A_0 is characterized by $\forall i(i \in A_0 \to i \in A)$.

If L_0 is recursively axiomatizable, then A_0 is recursively enumerable, so there is some recursive R such that $A_0 = \{i \in \omega : \exists j R(i, j)\}$. Then, in the defining formula, $i \in A_0$ can be replaced by $\exists j R(i, j)$. Similarly, if L_1 is decidable, $i \in A_1$ can be replaced by a recursive predicate $R'(i)$. These changes keep the formula Π^0_1 and eliminate the parameters. □

Thus, for any interval of modal or si-logics, the corresponding set of reals is $\boldsymbol{\Pi}^0_1$, in particular, it is a Borel set. It is a classical result in descriptive set theory that every Borel set has the perfect set property [7]. It follows that the cardinality of such a set is either countable or that of the continuum. As a result, we obtain the main theorem.

Theorem 1. *Let $L_1 \in \mathsf{NExt}L_0$ and $A_1 \subseteq Fml$ be the code of L_1. Then $[L_0, L_1]$ has the cardinality either countable or that of the continuum.*

The theorem applies to si-logics with straightforwardly adjusted proofs. The next corollary follows from the fact that $\mathsf{fmp}_{L_0}(L)$ always form an interval in transitive modal logics or si-logics, which was shown in [5]. This gives the solution to our problem.

Corollary 1.

1. *Let L_0 be a transitive modal logic, i.e., a normal modal logic containing $\boldsymbol{K4}$. Let $L \in \mathsf{NExt}L_0$. Then $\mathsf{fmp}_{L_0}(L)$ has the cardinality either countable or that of the continuum.*

2. *Let L be a si-logic. Then $\mathsf{fmp}(L)$ (in the lattice of si-logics) has the cardinality either countable or that of the continuum.*

In addition, we generalize the result to non-transitive modal logic L_0 by directly characterizing the complexity of $\mathsf{fmp}_{L_0}(L)$.

Lemma 2. *Let $L \in \mathsf{NExt}L_0$ with code A. Let $\mathsf{fmp}_{A_0}(A) = \{A' \subseteq Fml : L_{A'} \in \mathsf{NExt}L_0, \mathsf{FFr}(L_{A'}) = \mathsf{FFr}(L)\}$. Then $\mathsf{fmp}_{A_0}(A) \in \Pi^0_2(A_0 \oplus A)$. Moreover, if L_0 and L are recursively axiomatizable, then $\mathsf{fmp}_{A_0}(A) \in \Pi^0_2$.*

Proof Sketch. A finite Kripke frame is a finite set with a binary relation. So, finite Kripke frames (up to isomorphism) can be coded by natural numbers in an effective way, such that:

1. The set $\mathrm{FFr} = \{f \in \omega : f$ is a code of a finite Kripke frame$\}$ is recursive.

2. The validity relation $\mathrm{Val}(f, i)$ iff f is the code of a finite Kripke frame F and i is the code of a formula φ and $F \vDash \varphi$ is recursive.

This enables us to define the set $\mathsf{fmp}_{A_0}(A) \subseteq 2^\omega$ by a Π^0_2 formula with parameters A_0, A.

The second half of the statement follows by a similar argument in the proof of Lemma 1. □

A similar application of the perfect set property of Borel sets gives the next theorem.

Theorem 2. *Let L_0 be a normal modal logic. For any $L \in \mathsf{NExt}L_0$ the set $\mathsf{fmp}_{L_0}(L)$ has the cardinality either countable or that of the continuum.*

It is worth noting that these proofs do not use any special properties of modal or si-logics. Thus, this approach can be employed to investigate cardinalities of sets of other non-classical logical systems (with a reasonably simple syntax and semantics).

However, not all properties allow such straightforward characterization. A notable example is the degree of Kripke incompleteness. Although Blok [2] proved that the degree of Kripke incompleteness in $\mathsf{NExt}\mathbf{K}$ is either 1 or $2^{\aleph_0}$, the situation in other lattices, such as $\mathsf{NExt}\mathbf{K4}$ and $\mathsf{NExt}\mathbf{S4}$, remains unknown [3, Problem 10.5]. Given that all Kripke frames form a proper class, it becomes challenging to reason about Kripke frames using quantifiers over natural numbers or even reals, in contrast to finite Kripke frames.

We leave open the question of what implications the characterization within the Borel hierarchy may have for studying logical properties beyond the cardinality argument we presented. For example, if a logical property P is shown to be Borel, analytic, or belongs precisely to some complexity class C, what conclusions can we draw about that property in relation to logics?

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Generalization of terms up to equational theory

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A term s is a *generalization* of a term t if t can be obtained from s by variable substitution. The problem of identifying common generalizations for two or more terms has been the focus of substantial research, initiated in a series of papers by Plotkin [7], Popplestone [8], and Reynolds [9], all collected in the same volume published in 1970. The objective of these initial papers was to formalize an abstraction of the process of *inductive reasoning*. The main idea in this context is to find the solutions, i.e., the generalizing terms, that are *as close as possible* to the initial terms that define the problem. The existence and cardinality of this set of "best" solutions is encoded in what is called *generalization type*, which is a main object of study in this topic.

We provide a novel foundational approach to *equational* generalization, i.e., where terms are understood to be equivalent up to an equational theory [4]. The extension to generalization up to equational theories has been considered by several authors in theoretical computer science, and there is a growing interest in a general, foundational approach (see the recent survey [3]). We observe that relevant results so far have been obtained with ad hoc techniques developed for the specific equational theory under consideration (see e.g. results on semirings [1] and idempotent operations [2]).

The collection of methods developed to compute solutions to a generalization problem often goes under the name of *anti-unification*. This terminology suggests a connection between generalization and the arguably better known *unification* problems, where one seeks common instantiations to pairs of given terms. Our approach is indeed inspired by Ghilardi's algebraic setting for the study of equational unification problems [5].

Generally speaking, our methods are those of universal algebra, which is a most natural environment to handle equational theories from the side of their classes of models, i.e., *varieties*. In more detail, we first introduce a purely algebraic representation of equational generalization problems (called *e-generalization problems* from now on) and their solutions; secondly, we develop a universal-algebraic methodology for studying the e-generalization type, applying it in particular to (algebraizable) logics, where the considered equational theory is that of logical equivalence.

We show that e-generalization problems always have a best solutions (i.e., unitary type), in the following varieties: (abelian) groups, (commutative) semigroups and monoids; all varieties whose 1-generated free algebra is trivial, e.g., lattices, semilattices, varieties without constants whose operations are idempotent; Boolean algebras, Kleene algebras and Gödel algebras, which are the equivalent algebraic semantics of, respectively, classical, 3-valued Kleene, and Gödel-Dummett logic.

Symbolic e-generalization

The following definition of an e-generalization problem corresponds to the usual one used in the literature, just rephrased in the context of varieties and their free algebras; we only observe that while in the literature e-generalization problems are often considered to be just a pair of terms, we here consider the more general case of allowing a finite set of terms of any cardinality.

Definition 1. A *symbolic e-generalization problem* for a variety V is a finite set $\mathbf{t}$ of terms $t_1, \ldots, t_m \in \mathbf{F}_{\mathsf{V}}(X)$ for some finite set of variables X. A *solution* (or *generalizer*) is a term $s \in \mathbf{F}_{\mathsf{V}}(Y)$, with $Y = \mathrm{Var}(s)$, for which there exist substitutions $\sigma_1, \ldots, \sigma_m$ such that $\mathsf{V} \models \sigma_k(s) \approx t_k$ for all $k = 1, \ldots, m$. In this case we say that s is *witnessed* (or *testified*) by $\sigma_1, \ldots, \sigma_m$.

Any symbolic generalization problem $t_1, \ldots, t_m$ always has a solution: a fresh variable z, testified by the obvious substitutions $\sigma_k(z) = t_k$ for $k = 1, \ldots, m$. This is the *most general solution* for $t_1, \ldots, t_m$, in the sense that every other solution can be obtained from it by further substitution. In this context the interesting solutions are the *least general* ones, that are as *close* as possible to the initial terms representing the problem. Let us make these notions precise.

Consider two terms over the same language s and u; we say that s is *less general* than u, and write

$$s \preceq u, \text{ iff there exists a substitution } \sigma \text{ such that } \sigma(u) = s. \tag{0.1}$$

Let us then fix a problem $\mathbf{t} \subseteq \mathbf{F}_{\mathsf{V}}(X)$ and let $\mathcal{S}(\mathbf{t})$ be the set of its solutions; $\preceq$ is a preorder on $\mathcal{S}(\mathbf{t})$. With a slight abuse of notation we denote by $(\mathcal{S}(\mathbf{t}), \preceq)$ its associated poset of equally general solutions, that we call the *generality poset* of $\mathbf{t}$. Given a symbolic e-generalization problem $\mathbf{t}$, its *e-generalization type* is either: unitary, finitary, infinitary, or nullary, depending on the cardinality of any minimal (complete) set of solutions in $(\mathcal{S}(\mathbf{t}), \preceq)$.

Given a variety V, its symbolic e-generalization type is the worst possible type occurring among all its e-generalization problems, the best-to-worst order being: unitary > finitary > infinitary > nullary.

Algebraic e-generalization

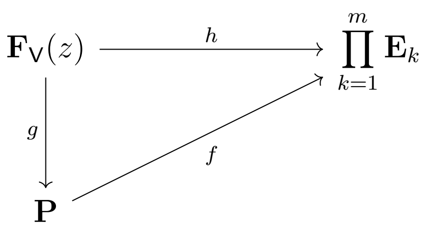
The algebraic translation of e-generalization problems uses projective and exact algebras in the considered variety. Let us call an algebra $\mathbf{A}$ a *retract* of an algebra $\mathbf{F}$ if there are homomorphisms $i : \mathbf{A} \to \mathbf{F}, j : \mathbf{F} \to \mathbf{A}$ such that $j \circ i = id_{\mathbf{A}}$ (and then necessarily i is injective and j is surjective); in this case we say that $\mathbf{A}$ is an (i, j)-retract of $\mathbf{F}$.

Consider a variety V. An algebra $\mathbf{P} \in \mathsf{V}$ is *projective in* V if it is a retract of a free algebra in V; an algebra $\mathbf{E} \in \mathsf{V}$ is called *exact in* V if it is isomorphic to a finitely generated subalgebra of some (finitely generated) free algebra. Evocatively, if we consider an exact algebra that is (isomorphic to) a subalgebra of a free algebra $\mathbf{F}_{\mathsf{V}}(X)$ generated by a term t, we write it as $\mathbf{E}(t)$.

The key idea that makes our translation works is to see the terms $t_1, \ldots, t_m$ representing the problem as a *single element* $(t_1, \ldots, t_m)$ of the direct product of the exact algebras $\mathbf{E}(t_k)$; it will be convenient to represent this tuple as the image of a fresh variable z via some map, which extends to a homomorphism on the 1-generated algebra in the associated variety, $\mathbf{F}_{\mathsf{V}}(z)$.

Definition 2. We call an *algebraic e-generalization problem* for a variety V a homomorphism $h : \mathbf{F}_{\mathsf{V}}(z) \longrightarrow \prod_{k=1}^{m} \mathbf{E}_k$ for some $m \geqslant 1$, where each $\mathbf{E}_k$ is an exact algebra in V.

A *solution* (or *generalizer*) for h is any homomorphism $g : \mathbf{F}_{\mathsf{V}}(z) \longrightarrow \mathbf{P}$, where $\mathbf{P}$ is finitely generated and projective in V, for which there exists a homomorphism $f : \mathbf{P} \longrightarrow \prod_{k=1}^{m} \mathbf{E}_k$ such that $f \circ g = h$, as illustrated in the following diagram:



We say that f *witnesses* or *testifies* the solution g.

Let us define a generality order among algebraic solutions. Fix an algebraic problem $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$. Given two generalizers $g : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}$, $g' : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}'$, we say that g is *less general* than g' and we write

$$g \sqsubseteq g' \text{ if and only if there exists } h : \mathbf{P}' \to \mathbf{P} \text{ such that } h \circ g' = g. \qquad (0.2)$$

The relation $\sqsubseteq$ is easily checked to be a preorder on the set of generalizers for h. We write $(\mathcal{A}(h), \sqsubseteq)$ for the corresponding poset of equally general generalizers. The *algebraic e-generalization type* of a problem is then given in complete analogy with the symbolic case, by checking the cardinality of a minimal complete set of solutions; similarly, one can define the algebraic e-generalization type of a variety as the worst possible type of its problems.

Let us now discuss how to translate back and forth between symbolic and algebraic problems and solutions. Let $\mathbf{t} = \{t_1, \ldots, t_m\} \subseteq \mathbf{F}_{\mathsf{V}}(X)$ be a symbolic e-generalization problem for a variety V, and $s \in \mathbf{F}_{\mathsf{V}}(Y)$ be a solution. Let us define $\mathsf{Alg}(\mathbf{t})$ and $\mathsf{Alg}(s)$ as the (unique) homomorphisms extending the following assignments:

$$\begin{aligned}\mathsf{Alg}(\mathbf{t}) : z \in \mathbf{F}_{\mathsf{V}}(z) &\longmapsto (t_1, \ldots, t_m) \in \prod_{k=1}^{m} \mathbf{E}(t_k),\\ \mathsf{Alg}(s) : z \in \mathbf{F}_{\mathsf{V}}(z) &\longmapsto s \in \mathbf{F}_{\mathsf{V}}(Y).\end{aligned}$$

Conversely, let $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$ be an algebraic e-generalization problem, where each $\mathbf{E}_k$ embeds via a homomorphism e_k to some free algebra $\mathbf{F}_{\mathsf{V}}(X_k)$; consider a solution $g : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}$, where $\mathbf{P}$ is an (i, j)-retract of $\mathbf{F}_{\mathsf{V}}(Y)$. Let p_k be the k-th projection on $\prod_{k=1}^{m} \mathbf{E}_k$, we define:

$$\begin{aligned}\mathsf{Sym}(h) &= \{t_1, \ldots, t_m\}, \quad \text{where } t_k = e_k \circ p_k \circ h(z) \text{ for } k = 1, \ldots m;\\ \mathsf{Sym}(g) &= (i \circ g)(z) \in \mathbf{F}_{\mathsf{V}}(Y).\end{aligned}$$

We can prove the following.

Theorem 1. *A symbolic e-generalization problem* $\mathbf{t} \subseteq \mathbf{F}_{\mathsf{V}}(X)$ *has a term* $s \in \mathbf{F}_{\mathsf{V}}(Y)$ *as solution if and only if* $\mathsf{Alg}(s)$ *is a solution to* $\mathsf{Alg}(\mathbf{t})$*; conversely, an algebraic e-generalization problem* $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$ *has a homomorphism* $g : \mathbf{F}_{\mathsf{V}}(z) \to \mathbf{P}$ *as a solution if and only if* $\mathsf{Sym}(g)$ *is a solution to* $\mathsf{Sym}(h)$*.*

Corollary 1. *Given a variety* V*, its symbolic and algebraic e-generalization types coincide.*

E-generalization type via congruences

After developing the general theory, we take advantage of some basic universal-algebraic tools, and develop a methodology based on the study of the congruence lattice of the 1-generated free algebra in the considered variety. In particular, we identify a class of varieties where the study of the generality type can be fully reduced to the study of this congruence lattice. Let us first transfer the notions of projective and exact from algebras to congruences: we call a congruence θ of a free algebra $\mathbf{F}_{\mathsf{V}}(X)$ *projective* (or *exact*) if $\mathbf{F}_{\mathsf{V}}(X)/\theta$ is projective (or exact) in V.

Definition 3. We say that an algebra $\mathbf{S}$ in a variety V is *strongly projective* in V if it is projective in V, and whenever there is an embedding $i : \mathbf{S} \to \mathbf{P}$, for some projective algebra $\mathbf{P}$, there is a homomorphism $j : \mathbf{P} \to \mathbf{S}$ such that $j \circ i = id_{\mathbf{S}}$. We say that a variety V is 1ESP if all 1-generated exact algebras in V are strongly projective in V.

Given any e-generalization problem h, let us call $\mathcal{G}(h)$ the set of all congruences that appear as kernels of its solutions; in other words, $\mathcal{G}(h)$ is the image under the function ker of the set $\mathcal{A}(h)$, given by all solutions to h: $\mathcal{G}(h) = \ker[\mathcal{A}(h)]$. In varieties such that every 1-generated exact algebra is (strongly) projective, one can show that:

$$\mathcal{G}(h) = \{\theta \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}}(z)) : \theta \subseteq \ker(h), \theta \text{ projective in } \mathsf{V}\}.$$

Theorem 2. *Let V be a 1ESP variety, and consider an algebraic e-generalization problem h. Its poset of solutions $\mathcal{A}(h)$ is dually isomorphic to the poset of congruences in $\mathcal{G}(h)$.*

Using this theorem, one can show that both Boolean and Kleene algebras have unitary e-generalization type, as well as all varieties whose 1-generated free algebra is trivial, e.g., lattices, semilattices, varieties without constants whose operations are idempotent.

Finally, we identify a sufficient condition for a problem, and for a variety, to have unitary e-generalization type.

Theorem 3. *Let $h : \mathbf{F}_{\mathsf{V}}(z) \to \prod_{k=1}^{m} \mathbf{E}_k$ be an algebraic e-generalization problem. If $\ker(h)$ is projective then the e-generalization type of h is unitary.*

Corollary 2. *If V is a variety for which finite intersections of exact congruences of $\mathbf{F}_{\mathsf{V}}(z)$ are projective, then V has unitary e-generalization type.*

As a consequence, one can see that the following varieties have unitary e-generalization type: abelian groups, commutative semigroups and monoids, Gödel algebras.

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Canonical model construction for a real-valued modal logic

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Introduction. Abelian logic, introduced in [2, 6], is the logic of lattice-ordered abelian groups, or, equivalently, $\mathbb{R}$ equipped with the operations min, max, $+$, $-$, and 0. A minimal modal extension of this logic, called Abelian modal logic, was defined in [3] based on standard Kripke frames, where the operations on $\mathbb{R}$ are calculated locally at worlds and the modal operator $\Box$ is interpreted by taking infima of values at accessible worlds. Abelian modal logic not only provides a framework for reasoning about (transitions between) states represented by vectors over $\mathbb{R}$, but also contains, under translation, the minimal Łukasiewicz modal logic studied in [4].

Notably, both Abelian modal logic and Łukasiewicz modal logic lack an explicit finitary axiomatisation. As a first step towards addressing this gap, an axiomatization was obtained in [3] for the modal-multiplicative fragment of the logic following a proof-theoretic approach. In [5], this approach was used to obtain a quasi-equational axiomatization of the equational theory of the modal-meet-semilattice-ordered-monoid fragment. In this work, we employ a canonical model construction to establish completeness of a quasi-equational axiomatization for a variation on the latter logic. In particular, instead of considering truth values in $\mathbb{R}$, we restrict our attention to the set $\mathbb{R}_{\leqslant 0}$ of non-negative real numbers, where 0 is considered the designated truth value, and the strictly negative numbers represent increasing degrees of falsehood. Alternatively, truth values can be taken from the open-closed unit interval $(0, 1]$, with the operations min, $\cdot$ (multiplication), and 1.

Semantics. Formulas $\varphi, \psi, \chi, \ldots$ are defined over a countably infinite set of propositional variables P with respect to a language with binary operation symbols $\wedge$ and $\oplus$, unary operation symbol $\Box$, and constant symbol e. We also define $0\varphi := \mathrm{e}$ and $(n+1)\varphi := n\varphi \oplus \varphi$ for $n \in \mathbb{N}$. An equation is an ordered pair of formulas, written $\varphi \approx \psi$, and $\varphi \leq \psi$ abbreviates $\varphi \wedge \psi \approx \varphi$.

Let $\mathbf{R}_{\leqslant 0}$ be the algebra $\langle \mathbb{R}_{\leqslant 0}, \min, +, 0\rangle$. A $\mathsf{K}(\mathbf{R}_{\leqslant 0})$-model $\mathfrak{M} = \langle W, R, V\rangle$ consists of a non-empty set of worlds W, an accessibility relation $R \subseteq W^2$, and an evaluation map V that assigns to each $p \in P$ a function $V(p)\colon W \to \mathbb{R}_{\leqslant 0}$. The value $[\![\varphi]\!](w)$ of a formula φ in $\mathfrak{M}$ at a world $w \in W$ is defined recursively as follows:

$$\begin{aligned}
[\![p]\!](w) &= V(p)(w)\\
[\![\mathrm{e}]\!](w) &= 0\\
[\![\psi \oplus \chi]\!](w) &= [\![\psi]\!](w) + [\![\chi]\!](w)\\
[\![\psi \wedge \chi]\!](w) &= \min([\![\psi]\!](w), [\![\chi]\!](w))\\
[\![\Box\psi]\!](w) &= \bigwedge\{[\![\psi]\!](v) \mid wRv\}.
\end{aligned}$$

Note that the meet in the interpretation of boxed formulas does not exist when the set contains arbitrarily small negative real numbers, and we therefore restrict our attention to models where this issue does not occur, i.e., models such that $[\![\varphi]\!](w)$ is defined for every formula φ and world $w \in W$. Note also that the empty meet is well-defined and equal to 0.

For an equation $\varphi \approx \psi$, we define

$$\models_{\mathsf{K}(\mathbf{R}_{\leqslant 0})} \varphi \approx \psi \quad :\Longleftrightarrow \quad [\![\varphi]\!](w) = [\![\psi]\!](w) \text{ for every } \mathsf{K}(\mathbf{R}_{\leqslant 0})\text{-model } \langle W, R, V\rangle \text{ and } w \in W$$

It is easily proved, along the same lines as the proof for Abelian modal logic in [3], that this logic admits a finite model property; that is, $\models_{\mathsf{K}(\mathbf{R}_{\leqslant 0})} \varphi \approx \psi$ if and only if $[\![\varphi]\!](w) = [\![\psi]\!](w)$ for every *finite* $\mathsf{K}(\mathbf{R}_{\leqslant 0})$-model $\langle W, R, V\rangle$ and $w \in W$. Decidability is also easily established, e.g., by providing a tableau proof system, again following the methodology of [3].

Axiomatisation. Next we define the target axiomatisation for our modal logic. Let $\mathcal{Q}_{\mathrm{msm}}$ denote the quasi-variety of algebras $\langle A, \wedge, \oplus, \mathrm{e}, \Box\rangle$ defined by equations axiomatizing the variety of meet-semilattice-ordered commutative monoids, together with the (quasi-)equations

- $x \leqslant \mathrm{e}$ (e is the greatest element);
- $x \oplus z \leqslant y \oplus z \implies x \leqslant y$ (cancellation);
- $nx \leqslant ny \implies x \leqslant y$ for each $n \in \mathbb{N}^+$ (torsion-freeness);
- $\Box x \oplus \Box y \leqslant \Box(x \oplus y)$;
- $\Box(nx) \approx n(\Box x)$ for each $n \in \mathbb{N}$.

We prove the following:

Theorem 1 (Completeness theorem)**.** *For any equation $\varphi \approx \psi$,*

$$\models_{\mathsf{K}(\mathbf{R}_{\leqslant 0})} \varphi \approx \psi \quad \Longleftrightarrow \quad \mathcal{Q}_{\mathrm{msm}} \models \varphi \approx \psi.$$

Canonical model construction. Recall that in classical modal logic the canonical model has as its worlds maximally consistent sets of formulas, or, equivalently, ultrafilters of the free Boolean algebra [1]. These are in one-to-one correspondence with homomorphisms from the free Boolean algebra to the two-element Boolean algebra. Let $\mathbf{F}_\Box$ denote the free algebra of $\mathscr{Q}_{\mathrm{msm}}$ over the set of generators P, and $\mathbf{F}$ its non-modal reduct. Analogously to the classical case, we define the set of worlds of our canonical model to be the set of homomorphisms from $\mathbf{F}$ to $\mathbf{R}_{\leqslant 0}$. We define an accessibility relation R on this set as follows:

$$h_1 R h_2 \quad :\iff \quad h_1(\Box a) \leqslant h_2(a) \quad \text{for each } a \in F,$$

i.e., for h_2 to be a successor of h_1 we request that h_2 internally "thinks" each a to be more true than what h_1 "thinks" about the boxed version $\Box a$. The valuation is standard: the value of $p \in P$ in a world h is defined to be $h(p)$; that is, $V(p)(h) := h(p)$.

We need two main results in order to establish the completeness theorem: a truth lemma and what we will call a separation lemma. As usual, the truth lemma states that the values a world h internally assigns to elements of $\mathbf{F}$ (essentially formulas), equals the "external" truth-value of the associated formulas in that world h. The separation lemma resembles the Lindenbaum lemma: given two distinct elements of $\mathbf{F}$ (essentially two non-equivalent formulas), we need to produce a world h that separates these, i.e., assigns them distinct values.

In both cases, we construct a homomorphism $h\colon \mathbf{F} \to \mathbf{R}_{\leqslant 0}$ by first defining a homomorphism g from a certain freely generated non-modal algebra $\mathbf{A}$ to $\mathbf{R}_{\leqslant 0}$. This $\mathbf{A}$ is defined in such a way that it admits $\mathbf{F}$ as a quotient. By ensuring that g factors through this quotient map, we find the desired homomorphism h. The proof that there exists a morphism g with the required properties makes use of Farkas' lemma from linear algebra. This is combined with topological techniques to reduce infinite sets of requirements to finite ones that differ between the two lemmata.

Lemma 1 (Separation lemma)**.** *Let $a, b \in \mathbf{F}_\Box$ with $a \not\leqslant b$. Then there exists a homomorphism $h\colon \mathbf{F} \to \mathbf{R}_{\leqslant 0}$ such that $h(a) > h(b)$.*

In particular, if $a \neq b$, then there exists a homomorphism $h\colon \mathbf{F} \to \mathbf{R}_{\leqslant 0}$ such that $h(a) \neq h(b)$.

Lemma 2 (Truth lemma)**.** *Let h be a world in the canonical model, and φ a formula. Then $[\![\varphi]\!](h) = (h \circ q)(\varphi)$, where q denotes the natural quotient-map from the set of formulas to $\mathbf{F}$.*

As usual, the proof of the Truth lemma proceeds by induction on φ. The cases for e, $\wedge$, and $\oplus$ are routine, and for $\Box$ the inequality $[\![\Box\varphi]\!](h) \leqslant (h \circ q)(\Box\varphi)$ follows from the definition of R. For the converse, we need to find a witness h' that is a successor of h such that $(h' \circ q)(\varphi) = (h \circ q)(\Box\varphi)$. The requirement that h' is a successor of h amounts to the requirement that $h(\Box a) \leqslant h'(a)$ for all a, so we find an infinite system of requirements. Using a topological compactness argument we reduce this to finitely many requirements, so that we can employ Farkas' lemma in order to find the desired h'.

Together, the Separation lemma and Truth lemma imply that $\mathbf{F}_\Box$ embeds into the complex algebra of the canonical model. Hence, our construction can be seen as a representation theorem, albeit only for the free algebra $\mathbf{F}_\Box$. In particular, this shows that any generalised quasi-equation valid in the complex algebra of the canonical model gives rise to an admissible rule in the logic. Restricting attention to equations, the completeness part of the Completeness theorem follows, i.e., $\models_{\mathsf{K}(\mathbf{R}_{\leqslant 0})} \varphi \approx \psi$ implies $\mathscr{Q}_{\mathrm{msm}} \models \varphi \approx \psi$. For soundness, it is easy to check that any complex algebra satisfies all the quasi-equations in the axiomatisation.

Future work. The results presented here represent a first step in an effort to obtain canonical models for real-valued modal logics, in particular Abelian modal logic and Łukasiewicz modal logic, with a view to establishing completeness results for suitable axiomatizations. A further step towards this goal would be to investigate either the logic considered in this work, or the corresponding fragment of Abelian modal logic studied in [5], extended with the binary join operator $\vee$ (interpreted as max in $\mathbb{R}$). However, it is currently unclear how to adapt the applications of Farkas' lemma in our proofs to deal with joins.

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On some properties of Płonka sums

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Introduction

The Płonka sum is a construction introduced in the 1960s in Universal Algebra by the eponymous Polish mathematician [7] (see also [9, 1]) that allows to construct a new algebra out of a semilattice direct system of similar (disjoint) algebras, called the fibers (of the system). The theory of Płonka sums has been mostly studied in the case of a similarity type without constant functional symbols: in such a case the fibres are subalgebras of their Płonka sum.

Płonka sums are strictly connected with regular identities. Recall that an identity $\alpha \approx \beta$ (in an algebraic language τ and over some set of variables X) is *regular* if $Var(\alpha) = Var(\beta)$. An identity $\alpha \approx \beta$ is valid in the Płonka sum over a non-trivial semilattice direct system $\mathbb{A} = ((\mathbf{A}_i)_{i\in I}, (I, \leq), (p_{ij})_{i\leq j})$ (i.e. $|I| \geq 2$) if and only if it is a regular identity valid in each of the fibers of $\mathbb{A}$.

Given a class of similar algebras $\mathscr{K}$, its *regularization* is the variety $\mathscr{R}(\mathscr{K})$ defined by the regular identities valid in $\mathscr{K}$. This variety is particularly interesting when the class $\mathscr{K}$ is a strongly irregular τ-variety $\mathscr{V}$ - an assumption that includes almost all examples of known irregular varieties -, i.e. a variety satisfying an identity of the form $p(x, y) \approx x$ for some binary τ-term p: in such a case, every algebra in $\mathscr{R}(\mathscr{V})$ is the Płonka sum over a semilattice direct systems (with zero) of algebras in $\mathscr{V}$.

The following is quite natural.

Question: which algebraic properties holding for $\mathscr{V}$ are also valid for $\mathscr{R}(\mathscr{V})$?

With respect to the question, several properties for $\mathscr{R}(\mathscr{V})$ have been established over the years, including the description of the lattice of the subvarieties of regularized varieties [3], of their subdirectly irreducible members [5], the equational basis of regularized varieties [11], and the structure of free algebras [10].

In this talk, we will give a brief overview of the theory of Płonka sums over an algebraic language *with constant symbols* with a particular emphasis on the structural side. Then we will address the above question with respect to the following properties: local finiteness, epimorphism surjectivity (ES), amalgamation property (AP) and congruence extension property (CEP).

Free algebras and local finiteness

Free algebras in the regularization $R(\mathscr{V})$ (of a strongly irregular variety $\mathscr{V}$) are characterized by Romanowska in [10] under the assumption that the language of $\mathscr{V}$ contains no constant symbols (see also [8]). The following covers the more general case of an algebraic language containing constants' symbols.

Theorem 1. *Let $\mathbf{A}$ be an algebra in $\mathscr{R}(\mathscr{V})$, with $\mathscr{V}$ a strongly irregular variety. Then $\mathbf{A} \in \mathscr{R}(\mathscr{V})$ is free on the set of generators $\{a_j\}_{j\in J}$ iff $\mathbf{A}$ is a Płonka sum over a semilattice direct sistem with zero $((\mathbf{A}_i)_{i\in I}, (I, \leq), (p_{ij})_{i\leq j})$ such that*

1. *$(I, \leq)$ is the free semilattice with zero on the set of generators J;*
2. *for every $i \in I, \mathbf{A}_i$ is a finitely generated $\mathscr{V}$-free algebra. In particular $\mathbf{A}_i$ has n generators $G^i = \{g^i_1, \dots, g^i_n\}$ iff $i = i_1 \vee \cdots \vee i_n$, for $i_1, \dots, i_n \in J$ ($i_k \neq i_m$ for $k \neq m$), and if $i \in J$ then $\mathbf{A}_i$ is one generated by the element $g^i_1 = a_{i_1} = a_i$;*
3. *for every $i \in I$, $p_{i_0 i}$ is the unique homomorphism from $\mathbf{A}_{i_0}$ into $\mathbf{A}_i$, where i_0 is the least element of I, while for every $i_1, ..., i_n \in J$ ($\forall n \in \mathbb{N}^+$) if $i = i_1 \vee ... \vee ... \vee i_n$ then for every $j \in \{1, ..., n\}$: $p_{i_j i} : \mathbf{A}_{i_j} \to \mathbf{A}_i$ is the unique (injective) homomorphism extending the map $p^0_{i_j i}\colon \{a_{i_j}\} \to \mathbf{A}_i$, $a_{i_j} \mapsto p^0_{i_j i}(g^i_j) := g^k_j$. In particular, for $i_0 \neq i \leqslant k$, $p_{ik}\colon \mathbf{A}_i \to \mathbf{A}_k$ is the unique (injective) homomorphism extending the map $p^0_{ik}\colon G^i \to \mathbf{A}_k$, $g^i_j \mapsto p^0_{ik}(g^i_j) := g^k_j$.*

Corollary 1. *Let $\mathscr{V}$ be a strongly irregular variety, $\mathscr{R}(\mathscr{V})$ its regularization, $\mathbf{A}_n$ the $\mathscr{R}(\mathscr{V})$-free algebra on $n \in \mathbb{N}$ generators, then*

$$|A_n| = \sum_{j=0}^{n} \binom{n}{j} |B_j|,$$

where $\mathbf{B}_j$ is the $\mathscr{V}$-free algebra on j generators.

Since local finiteness can be controlled on free algebras [2, Theorem 10.15], the following corollary holds.

Corollary 2. *Let $\mathscr{V}$ be a strongly irregular variety. If $\mathscr{V}$ is locally finite, then $\mathscr{R}(\mathscr{V})$ is locally finite.*

Congruences

In [1] a very natural **problem** is posed: to describe the congruence lattice of algebras in regular varieties.

Let $\mathbb{A} = ((\mathbf{A}_i)_{i\in I}, (I, \leq), (p_{ij})_{i\leq j})$ be a semilattice direct system in a strongly irregular τ-variety $\mathscr{V}$ and $\mathbf{A}$ its Płonka sum. Let's begin our investigation by starting with a congruence and trying to deduce its essential structural features.

Let $\theta \in Con(\mathbf{A})$, then for every $(i, j) \in I\times$ I we define $\theta_{ij} := \theta \cap (A_i \times A_j)$ and $S_\theta := \{(i, j) \in I \times I \mid \theta_{ij} \neq \varnothing\}$.

Lemma 1. *Let τ be any algebraic language, then $\forall(i, j) \in S_\theta, \forall a \in A_i$: $(a, p_{ii\vee j}(a)) \in \theta$. Moreover, S_θ is a reflexive and symmetric subsemilattice of $\mathbf{I} \times \mathbf{I}$.*

Unfortunately, *transitivity is not guaranteed*, but a (kind of) *weak* form of *transitivity*, outlined in the following Lemma, is always valid.

Lemma 2. *Let τ be any algebraic language, then $\forall i, j, k \in I : (i, j), (j, k) \in S_\theta \Rightarrow (i, i \vee k) \in S_\theta$.*

To simplify the exposition, we will say that S_θ is **upper transitive**.

In some particular, yet relevant, cases, S_θ turns out to be a congruence on $\mathbf{I}$.

Corollary 3. *Let τ be any algebraic language. If one of the following occurs:*

(i) $\mathbf{I}$ *is a chain;*

(ii) τ *be an algebraic language containing constants*

then $S_\theta \in Con(\mathbf{I})$.

Consequently, transitivity is always ensured for algebraic languages having constants.

The following result provides the sought-after characterization.

Theorem 2. *Let τ be any algebraic language. Let $S \subseteq I \times I$ and $(\theta_{ii})_{i\in I}$ be a family such that the following conditions occur:*

(i) S is a reflexive, symmetric and upper transitive subsemilattice of $\mathbf{I} \times \mathbf{I}$;

(ii) $\forall i \in I : \theta_{ii} \in Con(\mathbf{A}_i)$;

(iii) $\forall (i,j) \in I \times I : \theta_{ii} \subseteq (p_{ii\vee j} \times p_{ii\vee j})^{-1}(\theta_{i\vee j, i\vee j})$, *with equality if* $(i,j) \in S$;

(iv) $\forall (i,j) \in I \times I : (i,j) \in S \iff (i, i \vee j), (j, i \vee j) \in S, (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j}) \neq \varnothing$

For every $(i,j) \in S \setminus \Delta_{\mathbf{I}}$, *let* $\theta_{ij} := (p_{ii\vee j} \times p_{ji\vee j})^{-1}(\theta_{i\vee j, i\vee j})$, *then*

$$\theta := \bigcup_{(i,j)\in S} \theta_{ij} \in Con(\mathbf{A}).$$

Furthermore, all the elements of $Con(\mathbf{A})$ *arise in this way.*

In the case of an algebraic language containing constants (or if $\mathscr{V}$ admits an algebraic constant), the characterization takes on a simpler form, since requirement *(iv)* is automatically satisfied and $S \in Con(\mathbf{I})$.

Amalgamation and Congruence Extension Property

It is very natural to ask whether the amalgamation property (AP) can be "lifted" through Płonka sums. More precisely, does $\mathscr{R}(\mathscr{V})$ have the (strong) AP when $\mathscr{V}$ is strongly irregular and has (strong) AP?

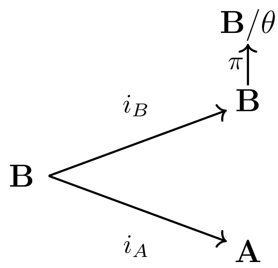
The fact that semilattices (with zero) have (strong) AP could point to a positive answer. Surprisingly enough, Hall [4, Remark 5] showed that Clifford semigroups, namely the regularization of groups (see [9] for details), fail to have AP.

Thanks to the following notion, Hall's argument can be easily generalized.

Definition 1. An algebra $\mathbf{A}$ is *hereditarily simple* if each of its subalgebras is simple.
A variety $\mathscr{V}$ is *hereditarily simple* if each simple algebra in $\mathscr{V}$ is hereditarily simple.

The following result by Pastijn [6] links the validity of the congruence extension property in a strongly irregular variety to the existence of amalgams in $\mathscr{R}(\mathscr{V})$ for specific V-formations in $\mathscr{R}(\mathscr{V})$.

Proposition 1 ([6]). *Let* $\mathscr{V}$ *be a strongly irregular variety. Then* $\mathscr{V}$ *has CEP iff* $\forall \mathbf{A} \in \mathscr{V}, \forall \mathbf{B} \leq \mathbf{A}, \forall \theta \in Con(\mathbf{B})$ *the following* V*-formation in* $\mathscr{R}(\mathscr{V})$



has an amalgam in $\mathscr{R}(\mathscr{V})$*.*

Corollary 4. *Let* $\mathscr{V}$ *be a strongly irregular variety. If* $\mathscr{R}(\mathscr{V})$ *has* AP*, then* $\mathscr{V}$ *has* CEP*. In particular, if* $\mathscr{V}$ *is not hereditarily simple, then* $\mathscr{R}(\mathscr{V})$ *fails to have* AP*.*

Pastijn [6] indeed gives an answer to our question with respect to (strong) AP and CEP.

Theorem 3 ([6])**.** *Let* $\mathscr{V}$ *be a strongly irregular variety. Then:*

1. $\mathscr{R}(\mathscr{V})$ *has* CEP *if and only if* $\mathscr{V}$ *has* CEP*.*
2. $\mathscr{R}(\mathscr{V})$ *has (strong)* AP *if and only if* $\mathscr{V}$ *has* CEP *and (strong)* AP*.*

Epimorphism Surjectivity

Epimorphism surjectivity is another property preserved by Płonka sums. More specifically

Theorem 4. *Let* $\mathscr{V}$ *be a strongly irregular variety.* $\mathscr{R}(\mathscr{V})$ *has* ES *if and only if* $\mathscr{V}$ *has* ES*.*

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