The Beth companion: making implicit operations explicit. Part I

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Let K be a class of algebras.

Definition 1. *An n-ary partial function on K is a tuple $\langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$, where each $f^{\mathbf{A}}$ is a function $f^{\mathbf{A}} \colon X \to A$ for some $X \subseteq A^n$. The set X is then called the* domain *of $f^{\mathbf{A}}$ and denoted with $\mathsf{dom}(f^{\mathbf{A}})$.*

We are interested in particular partial functions that exhibit a behaviour similar to that of term functions.

Definition 2. *Let $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$ be a partial function on K. Then f is* preserved by homomorphisms *when for every homomorphism $h \colon \mathbf{A} \to \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ and $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\mathbf{A}})$ we have $\langle h(a_1), \dots, h(a_n)\rangle \in \mathsf{dom}(f^{\mathbf{B}})$ and*

$$h(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(h(a_1), \dots, h(a_n)).$$

Definition 3. *A first-order formula φ in the language of K is said to be* functional *in K when for every $\mathbf{A} \in \mathsf{K}$ and $a_1, \dots, a_n \in A$ there exists at most one $b \in A$ such that $\mathbf{A} \vDash \varphi(a_1, \dots, a_n, b)$.*

A functional formula φ induces an n-ary partial function $\varphi^{\mathbf{A}}$ on each $\mathbf{A} \in \mathsf{K}$ with domain

$$\mathsf{dom}(\varphi^{\mathbf{A}}) = \{\langle a_1, \dots, a_n\rangle \in A^n : \text{there exists } b \in A \text{ such that } \mathbf{A} \vDash \varphi(a_1, \dots, a_n, b)\}$$

defined for every $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(\varphi^{\mathbf{A}})$ as $\varphi^{\mathbf{A}}(a_1, \dots, a_n) = b$, where b is the unique element of A such that $\mathbf{A} \vDash \varphi(a_1, \dots, a_n, b)$.

Definition 4. *A partial function $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$ on K is called* implicit *if there exists a functional formula φ such that $f^{\mathbf{A}} = \varphi^{\mathbf{A}}$ for each $\mathbf{A} \in \mathsf{K}$. In this case we say that φ defines f.*

Definition 5. *An* implicit operation *of K is an implicit partial function of K that, moreover, is preserved by homomorphisms.*

Example 6. Let DL be the variety of bounded distributive lattices and φ the formula

$$(x \wedge y \approx 0) \,\&\, (x \vee y \approx 1).$$

If $\mathbf{A} \in \mathsf{DL}$, then $\mathbf{A} \vDash \varphi(a, b)$ if and only if b is a complement of a in $\mathbf{A}$. Since complements in bounded distributive lattices are unique when they exist and are preserved by bounded lattice homomorphisms, we conclude that φ defines an implicit operation of DL.

[*] Speaker.

It is well known that the formulas preserved by homomorphisms are precisely the *existential positive formulas*; that is, the formulas built from equations and $\bot$ using only existential quantifiers, conjunctions, and disjunctions. As a consequence, we obtain the following characterization of implicit operations.

Proposition 7. *Let* K *be an elementary class and* f *a partial function on* K*. Then* f *is an implicit operation if and only if it is defined by an existential positive formula.*

Definition 8. *An* n*-ary implicit operation* f *is said to be* interpolated in K *by a set* $\mathcal{T}$ *of* n*-ary terms in the language of* K *when for every* $\mathbf{A} \in \mathsf{K}$ *and* $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f^{\mathbf{A}})$ *there exists* $t \in \mathcal{T}$ *such that*

$$f^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n).$$

Intuitively, the implicit operation f is made explicit by the terms in $\mathcal{T}$.

Example 9. Every term-function is clearly an implicit operation that is interpolated by a single term. However, not all implicit operations can be interpolated by terms. For instance, there is no set of terms in the language of bounded distributive lattices interpolating the operation of taking complements in DL.

Definition 10. *A class of algebras is said to have the* strong Beth definability property *when each of its implicit operations can be interpolated by a set of terms.*

The name "strong Beth definability property" is motivated by the resemblance to the *Beth definability property* in logic. When a finitary logic is algebraized by a quasivariety K, the former has the Beth definability property iff all epimorphisms in K are surjective [1, Thm. 3.12] (see also [2]). For quasivarities the strong Beth definability property can be conveniently phrased in the following equivalent form, where we recall that a *primitive positive formula* (*pp formula*, for short) is a conjunction of equations prenexed by existential quantifiers.

Proposition 11. *A quasivariety has the* strong Beth definability property *if and only if each of its implicit operations defined by pp formulas can be interpolated by a single term.*

The strong Beth definability property corresponds to a condition stronger than the surjectivity of epimorphisms (see [3, 4] for this correspondence in the case of modal and intuitionistic logic), which is defined as follows:

Definition 12. *A class of algebras* K *has the* strong epimorphism surjectivity property (SES property*, for short*) *when for every homomorphism* $f\colon \mathbf{A} \to \mathbf{B}$ *with* $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ *and* $b \in B - f[A]$ *there exists a pair of homomorphisms* $g, h\colon \mathbf{B} \to \mathbf{C}$ *with* $\mathbf{C} \in \mathsf{K}$ *such that* $g \circ f = h \circ f$ *and* $g(b) \neq h(b)$*.*

Theorem 13. *A universal class has the SES property iff it has the strong Beth definability property.*

When K is a quasivariety, the SES property can be formulated in purely categorical terms: K has the SES property iff all monomorphisms in K are regular.

Our main contribution is a method to optimally expand a given class of algebras into one where all implicit operations can be made explicit. The guiding example is the variety of Boolean algebras, which can be obtained by adding the operation of taking complements to the variety of bounded distributive lattices.

Let $\mathcal{F}$ be a set of implicit operations of K. We denote by $\mathscr{L}_{\mathcal{F}}$ the language obtained by expanding the language of K with function symbols acting as names for the operations in $\mathcal{F}$.

Whenever $\mathbf{A} \in \mathsf{K}$ has the property that $f^{\mathbf{A}}$ is a total function for each $f \in \mathcal{F}$, we can expand $\mathbf{A}$ to an algebra $\mathbf{A}[\mathscr{L}_{\mathcal{F}}]$ in the language $\mathscr{L}_{\mathcal{F}}$ by interpreting the new function symbols with the respective implicit operations in $\mathcal{F}$. We can then consider the collection of such expansions:

$$\mathsf{K}[\mathscr{L}_{\mathcal{F}}] \coloneqq \{\mathbf{A}[\mathscr{L}_{\mathcal{F}}] : \mathbf{A} \in \mathsf{K} \text{ and } f^{\mathbf{A}} \text{ is total for each } f \in \mathcal{F}\}.$$

Definition 14. *An implicit operation f of K is said to be* extendable *in K if every $\mathbf{A} \in \mathsf{K}$ can be embedded into some $\mathbf{B} \in \mathsf{K}$ such that $f^{\mathbf{B}}$ is a total function.*

Definition 15. *A* pp expansion M *of K is the class of subalgebras of $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ for some set $\mathcal{F}$ of extendable implicit operations of K that are defined by pp formulas. If moreover M has the strong Beth definability property, we call it a* Beth companion *of K.*

Examples 16.

(i) The variety of Boolean algebras is a Beth companion of the variety of bounded distributive lattices.

(ii) The variety of implicative semilattices is a Beth companion of the variety of Hilbert algebras.

(iii) The variety of Heyting algebras of depth at most 2 is a Beth companion of the variety of pseudocomplemented distributive lattices.

(iv) The variety of Abelian groups is a Beth companion of the quasivariety of cancellative commutative monoids.

But not every class of algebras has a Beth companion.

Examples 17.

(i) The variety of (commutative) monoids does not admit a Beth companion.

(ii) Infinitely many varieties of Heyting algebras do not admit a Beth companion. In particular, for every $n \geq 5$ the variety generated by the n-element Heyting chain lacks a Beth companion.

As a main result, we obtain that Beth companions have the following desirable properties:

Theorem 18.

(i) *A Beth companion of a quasivariety is a quasivariety.*

(ii) *All the Beth companions of a quasivariety are term-equivalent.*

References

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