Computing and formalizing residuated lattices and relation algebras

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Extended abstract

Much of my research has been inspired by the use of computers to look at finite examples of partially ordered algebras. Automated theorem provers like Prover9-Mace4 have also been used to sometimes find surprising proofs or counterexamples. Here I will indicate how such experiments can be used to investigate residuated lattices and related algebras.

Starting with software that produces small databases of finite algebras, I will present programs that construct and display congruence lattices, subalgebras, nuclear images, poset products and Płonka sums of finite residuated po-magmas and residuated lattices.

For relation algebras an interesting question is whether they are representable by binary relations. Robin Hirsch and Ian Hodkinson showed that even restricted to finite relation algebras this is an undecidable problem. But Roger Maddux's concept of n-dimensional bases and Steve Comer's one-point extension method can prove (non)representability for various small algebras. We revisit implementations of these algorithms and apply them to all relation algebras up to size 32. Both methods are based on a two-player game for representability, and in a paper with Jas Semrl we recently extended this approach to a representability game for weakening relation algebras. I will report on implementing this game and applying it to investigate the representability of small distributive involutive residuated lattices.

Lean is an interactive theorem prover that uses a formal language based on dependent type theory to represent mathematics. Its library of definitions and theorems spans many areas of mathematics, including parts of algebra, logic, order theory and category theory. In joint work with Pace Nelson we are developing theories of residuated po-magmas, residuated lattices and relation algebras in Lean. We note that Lean is also an efficient functional programming language, hence this is a promising platform for implementing provably correct algorithms for partially ordered algebras.

Since this installment of LATD is paying homage to Franco Montagna, I would like to conclude with the following reminiscence: I first met Franco at a conference in Brno, Czech Republic in October 2003. While traveling to the conference, I had computed all GBL-chains with up to 8 elements and noticed that they were all commutative. I mentioned this during my talk, and afterwards Franco asked if I would like to come to Siena the following year. So in May 2004, on my first visit to Italy, I got to do three weeks of research with Franco at the University of Siena. I stayed in the magical Certosa di Pontignano and took a 30-minute bus ride to/from Siena each morning and evening through the beautiful Tuscan landscape. After very interesting conversations with Franco each day, I distinctly remember having several surprising insights while looking absentmindedly out the bus windows into the spectacular countryside. And during the visit Franco and I proved that all finite GBL-algebras are indeed commutative. I am forever grateful to Franco to have given me the wonderful opportunity to know him and collaborate with him for more than a decade.