Amalgamation in classes of involutive commutative residuated lattices

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1 Introduction

Amalgamation is explored in this talk within classes of involutive commutative residuated lattices that are non-divisible, non-integral, and non-idempotent. Several classes of algebras significant to us are designated by a distinctive notation:

$\mathfrak{A}^{\mathfrak{c}}$ the class of abelian o-groups
$\mathfrak{I}$ the class of involutive FL_e-algebras
$\mathfrak{S}$ the class of odd or even idempotent-symmetric involutive FL_e-algebras

Adjunct to $\mathfrak{I}$,

- the superscript $\mathfrak{c}$ means restriction to totally-ordered algebras,
- the superscript $\mathfrak{sl}$ means restriction to semilinear algebras,
- the subscript $\mathfrak{o}$ means restriction to odd algebras,
- the subscript $\mathfrak{e}$ means restriction to even algebras,
- the subscript $\mathfrak{e}_{\mathfrak{i}}$ means restriction to even algebras having an idempotent falsum constant,
- the subscript $\mathfrak{e}_{\mathfrak{n}}$ means restriction to even algebras having a non-idempotent falsum constant,

When multiple letters appear in the subscript, they denote the union of the corresponding classes. For instance $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{o}\mathfrak{e}_{\mathfrak{i}}}$ refers to the class of idempotent-symmetric involutive FL_e-chains which are either odd or even with an idempotent falsum constant.

First we delve into the Amalgamation Property within subclasses of $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{o}\mathfrak{e}}$. We show that several subclasses of these structures fail to satisfy the Amalgamation Property (Theorem 1), including the classes of odd and even ones. This failure stems from the same underlying reason as in the case of discrete linearly ordered abelian groups with positive normal homomorphisms [3]. Conversely, it is proven that three subclasses of them exclusively comprising algebras that are idempotent-symmetric possess the Amalgamation Property (Theorem 2), albeit fail the Strong Amalgamation Property (Theorem 3). The failure of the Strong Amalgamation Property in these subclasses can be attributed to the same underlying reason observed in the class of linearly ordered abelian groups with positive homomorphisms [1].

Then we shift our focus from these classes of chains to the semilinear varieties of FL_e-algebras that they generate. Our goal is to transfer the Amalgamation Property, or its failure, from the specific classes of chains to the generated varieties. We conclude that every variety of semilinear involutive commutative (pointed) residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property (Theorem 4). This result strengthens a recent proof by W. Fussner and S. Santschi, which established that the variety of semilinear involutive commutative residuated lattices lacks the Amalgamation Property [2, Theorem 5.2]. Furthermore, we demonstrate that the varieties of idempotent-symmetric, semilinear, odd involutive residuated lattices, as well as idempotent-symmetric, semilinear, odd or even involutive residuated lattices, exhibit the Transferable Injections Property (Theorem 5), a strengthening of the Amalgamation Property.

2 Amalgamation in classes of $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{oe}}$

Theorem 1. *The classes $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{e}}$, $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{e}_{\mathfrak{i}}}$, $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{e}_{\mathfrak{n}}}$, along with every class of involutive FL_e-chains which contains $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{o}}$, fail the Amalgamation Property.*

Theorem 2. *The classes $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{o}}$, $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{e}}$, and $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{oe}}$ each satisfy the Amalgamation Property.*

Theorem 3. *The classes $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{o}}$, $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{e}}$, and $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{oe}}$ do not satisfy the Strong Amalgamation Property.*

3 Amalgamation in the generated semilinear varieties

Theorem 4. *Every variety of semilinear involutive commutative (pointed) residuated lattices that includes the variety of odd semilinear commutative residuated lattices fails the Amalgamation Property.*

Theorem 5. *The varieties $\mathfrak{S}^{\mathfrak{sl}}_{\mathfrak{o}}$ and $V(\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{e}})$ have the Transferable Injections Property.*

4 Techniques

The core principle of our approach relies on leveraging the intrinsic layer group decomposition of the algebras in $\mathfrak{I}^{\mathfrak{c}}_{\mathfrak{oe}}$ [4] and an associated categorical equivalence [5]. This strategic direct system decomposition facilitates the independent execution of amalgamation within each distinct layer. Subsequently, these layer-wise amalgams are leveraged to construct the overall amalgam of the algebras via the functor detailed in [5] (see Fig. 1).

As an example, proving Theorem 6 was necessary to convert the cyan direct system into the brown one. Additionally, several techniques for embedding direct systems into those over larger index sets were developed to construct the embeddings shown in Fig. 1.

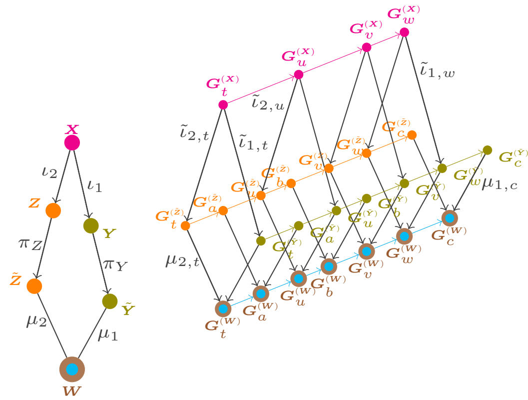

Fig. 1: Brief visual illustration of the main constructions: "Layerwise" amalgamation in $\mathfrak{A}^{\mathfrak{c}}$ (right), and the corresponding amalgamation in $\mathfrak{S}^{\mathfrak{c}}_{\mathfrak{oc}}$ (left).

Theorem 6. *For any direct system $\langle \boldsymbol{L}_u, \varsigma_{u \to v} \rangle_\kappa$ of torsion-free partially ordered abelian groups over an arbitrary chain κ, there exists a direct system $\langle \widehat{\boldsymbol{G}}_u, \varsigma_{u \to v} \rangle_\kappa$ of abelian o-groups. In this system the abelian group reducts of the $\boldsymbol{L}_u$'s and the transitions remain unchanged, while, for every $u \in \kappa$, the ordering relation of $\widehat{\boldsymbol{G}}_u$ is an extension of the ordering relation of $\boldsymbol{L}_u$.*

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