Semilinear Finitary Extensions of Pointed Abelian logic

Filip Jankovec

Institute of Computer Science of the Czech Academy of Sciences

Abstract

In this talk, we will consider extensions of pointed abelian logic determined by subquasivarieties of a class of pointed Abelian ℓ-groups. In particular, we will focus on those quasivarieties which are generated by chains.

Łukasiewicz logic in its infinitely-valued version was introduced by Łukasiewicz and Tarski [14] in 1930 and since then it was proved to be one of the most prominent non-classical logics. This logic is by itself a member of the family of many-valued logics often used to model some aspects of vagueness. Also, it has deep connections with other areas of mathematics such as continuous model theory, error-correcting codes, geometry, algebraic probability theory, etc. [3, 6, 9, 11].

Abelian logic is a well-known (finitary) contraclassical paraconsistent logic. This logic was independently introduced by Meyer and Slaney [10] and by Casari [2] and it is also called the logic of Abelian ℓ-groups [1] or Abelian Group Logic [12]. This terminology follows from the fact that the matrix models of Abelian logic consist of Abelian ℓ-groups and their positive cones as filters of designated elements (there is also a version of Abelian logic in which the only designated element is the neutral element of the group, which will not be considered here).

Pointed Abelian logic is expansion of Abelian logic, where we add a new constant symbol $\mathbf{f}$ to the language, but do not add any axioms. This new constant greatly improves the expressive power of our logic. In particular, this logic contains several important extensions, the most important of which was Łukasiewicz unbound logic (see[4]).

The varieties of MV-algebras, classified by Komori in [8], correspond to varieties of positively ($f \geq 0$) pointed abelian ℓ-groups, as shown by Young in [13] via the Mundici functor (for the definition of the Mundici functor, see [3]). We will generalize this to classification of all pointed ℓ-groups.

The whole variety of pointed Abelian ℓ-groups is generated by $\{\mathbf{R}_{-1}, \mathbf{R}_1\}$ and $\{\mathbf{Q}_{-1}, \mathbf{Q}_1\}$, respectively. The subvarieties of $\mathbf{HSP}(\mathbf{R}_1)$ are determined by algebras $\mathbf{Z}_n$ and $\mathbf{Z}_n \overrightarrow{\times} \mathbf{Z}_0$ for $n, > 0$ and subvarieties of $\mathbf{HSP}(\mathbf{R}_{-1})$ are determined by algebras $\mathbf{Z}_n$ and $\mathbf{Z}_n \overrightarrow{\times} \mathbf{Z}_0$ for $n, < 0$. This can be generalized to the description of all subvarieties of pointed Abelian ℓ-groups.

Our next goal is to generalize this classification to all quasivarieties generated by chains. The motivation for this approach is that these quasivarieties correspond to semilinear extensions of pointed abelian logic. In [7] Gispert described all semilinear finite extensions of Łukasiewicz logic by describing all universal classes of MV-chains, thus (using [5]) giving a classification of all quasi-varieties of MV-algebras generated by chains. We show that this classification can also be applied to pointed ℓ-groups by proving the following lemma.

Lemma 1. *Let $\boldsymbol{A}$ be an Abelian ℓ-group a let $\boldsymbol{B}$ be a convex subgroup of $\boldsymbol{A}$, and let be $b \in B$ a strong unit in $\boldsymbol{B}$. Then $\mathbf{ISPP_U}(\boldsymbol{A}_b) = \mathbf{ISPP_U}(\boldsymbol{B}_b)$.*

In other words, this lemma tells us that we can restrict ourselves to groups with a strong unit. These are known to be equivalent to MV-algebras via the Mundici functor. Therefore, we can describe all quasivarieties of pointed Abelian ℓ-groups generated by chains as it is stated in the following theorem.

Theorem 1. *Let* $\mathbf{S}$ *denote any finitely generated dense* ℓ*-subgroup of* $\mathbf{R}$ *such that* $\mathbf{S} \cap \mathbf{Q} = \mathbf{Z}$. *Every subquasivariety of pAb generated by chains is equal to*

$$\mathbf{ISPP_U}(\{\mathbf{Z_n} \mid \mathbf{n} \in \mathbf{A}\} \cup \{\mathbf{Z_n} \overrightarrow{\times} \mathbf{Z_m} \mid \mathbf{n} \in \mathbf{B}, \mathbf{m} \in \gamma(\mathbf{n}) \cup \{\mathbf{S_d} \mid \mathbf{d} \in \mathbf{C}\}\}),$$

for some $A, B, C \subseteq Z$, *and* $\gamma : n \mapsto \gamma(n) \subseteq \mathrm{div}(n)$, *where* $\mathrm{div(n)}$ *stands for the set of all divisors of* $n \in Z$.

Although the above result can be derived quite easily from [7] using our Lemma 1 and the Mundici functor, we try to prove these results without using theory of MV-algebras. We believe that this will lead to a significant simplification of the proofs used. In the last section we give an axiomatization of these quasivarieties.

References

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