The weak Robinson property

Isabel Hortelano Martín, George Metcalfe, and Simon Santschi

Mathematical Institute, University of Bern, Switzerland
{isabel.hortelanomartin,george.metcalfe,simon.santschi}@unibe.ch

The connection between the algebraic property of amalgamation and the syntactic property of interpolation has received considerable attention in the literature in the frameworks of model theory [1], abstract algebraic logic [2], universal algebra [5], and residuated structures [4, 6]. Explicitly, if a logic $\vdash$ is algebraized by a variety $\mathcal{V}$ that has the congruence extension property, then $\mathcal{V}$ has the amalgamation property if and only if $\vdash$ has the deductive interpolation property. This "bridge theorem" provides a powerful technique for establishing the deductive interpolation property via the amalgamation property, and vice versa. However, for varieties that lack the congruence extension property, failure of the amalgamation property does not necessarily imply failure of the deductive interpolation property. A natural problem is therefore to describe a property that, when combined with the deductive interpolation property, is equivalent to the amalgamation property. In this work, we identify such a property and provide an algebraic characterization, offering a potential pathway to resolving certain open problems in the area.

1 Amalgamation and interpolation

The term "amalgamation" refers to the process by which two algebras are combined while preserving a common subalgebra. To express this notion formally, let $\mathcal{K}$ be a class of similar algebras. A *doubly injective span* in $\mathcal{K}$ is a 5-tuple $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \varphi_B, \varphi_C \rangle$, consisting of algebras $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \boldsymbol{A} \to \boldsymbol{B}$ and $\varphi_C \colon \boldsymbol{A} \to \boldsymbol{C}$. The class $\mathcal{K}$ is said to have the *amalgamation property* (for short, AP) if for every doubly injective span $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \varphi_B, \varphi_C \rangle$ in $\mathcal{K}$, there exist an algebra $\boldsymbol{D} \in \mathcal{K}$, and embeddings $\psi_B \colon \boldsymbol{B} \to \boldsymbol{D}$ and $\psi_C \colon \boldsymbol{C} \to \boldsymbol{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.

The AP for a variety can be characterized in terms of its free algebras, which can be then reflected in a property of the corresponding equational consequence relation of the variety. In particular, we may focus on the equational consequence relation for a fixed countably infinite set of variables X. Formally, the consequence relation $\vDash_{\mathcal{K}}$ on the set of equations $Eq(X)$ (pairs of formulas over X) is defined as follows for $\Sigma \cup \{\varepsilon\} \subseteq Eq(X)$:

$$\Sigma \vDash_{\mathcal{V}} \varepsilon \ :\Longleftrightarrow \ \text{for any homomorphism } h \text{ from the formula algebra over } X \text{ to some } \boldsymbol{A} \in \mathcal{K}, \quad \Sigma \subseteq \ker(h) \implies \varepsilon \in \ker(h).$$

Given $\Sigma \cup \Gamma \subseteq Eq(X)$, we write $\Sigma \vDash_{\mathcal{V}} \Gamma$ if $\Sigma \vDash_{\mathcal{V}} \gamma$ for all $\gamma \in \Gamma$, and denote by $Var(\Gamma)$ the set of variables occurring in Γ. If $\mathcal{V}$ is a variety, then the equational consequence relation $\vDash_{\mathcal{V}}$ is finitary. Moreover, if $\vdash$ is an algebraizable logic with equivalent algebraic semantics $\mathcal{V}$, then there are mutually inverse translations between $\vdash$ and $\vDash_{\mathcal{V}}$.

A variety $\mathcal{V}$ has the *Robinson property* (for short, RP) if for any $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(X)$ such that $Var(\Sigma) \cap Var(\Pi) \neq \emptyset$ and $Var(\{\varepsilon\}) \cap Var(\Pi) \subseteq Var(\Sigma)$, whenever

(i) $\Sigma \vDash_{\mathcal{V}} \delta \Longleftrightarrow \Pi \vDash_{\mathcal{V}} \delta$ for all $\delta \in Eq(X)$ with $Var(\delta) \subseteq Var(\Sigma) \cap Var(\Pi)$;

(ii) $\Sigma \cup \Pi \vDash_{\mathcal{V}} \varepsilon$,

then $\Sigma \vDash_{\mathcal{V}} \varepsilon$.

Theorem 1 (cf. [5, Thm. 13])**.** *A variety has the amalgamation property if and only if it has the Robinson property.*

The RP (and hence the AP) implies the deductive interpolation property, whose algebraic counterpart is the "generalized amalgamation property with injections", introduced by Kihara and Ono in [4]. Formally, a variety $\mathcal{V}$ is said to have the *deductive interpolation property* (for short, DIP) if for any $\Sigma \cup \{\varepsilon\} \subseteq Eq(X)$ such that $Var(\Sigma) \cap Var(\{\varepsilon\}) \neq \emptyset$, whenever

(i) $\Sigma \models_{\mathcal{V}} \varepsilon$,

then there exists $\Delta \subseteq Eq(X)$ with $Var(\Delta) \subseteq Var(\Sigma) \cap Var(\{\varepsilon\})$ such that

(ii) $\Sigma \models_{\mathcal{V}} \Delta$;

(iii) $\Delta \models_{\mathcal{V}} \varepsilon$.

Conversely, the DIP implies the AP in the presence of the congruence extension property. Recall that a variety $\mathcal{V}$ has the *congruence extension property* (for short, CEP) if for every $\boldsymbol{D} \in \mathcal{V}$, subalgebra $\boldsymbol{C}$ of $\boldsymbol{D}$, and congruence Θ of $\boldsymbol{C}$, there exists a congruence Φ of $\boldsymbol{D}$ such that $\Theta = \Phi \cap C^2$.

The syntactic counterpart of the CEP is the extension property (see [6, Sec. 8.2]). A variety $\mathcal{V}$ is said to have the *extension property* (for short, EP) if for any $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(X)$, whenever

(i) $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$,

then there exists $\Delta \subseteq Eq(X)$ with $Var(\Delta) \subseteq Var(\Pi \cup \{\varepsilon\})$ such that

(ii) $\Sigma \models_{\mathcal{V}} \Delta$;

(iii) $\Delta \cup \Pi \models_{\mathcal{V}} \varepsilon$.

On the syntactic side, the connection between the RP and DIP can be made explicit in the form of the following theorem, which appeared first in [3].

Theorem 2 (cf. [5, Thm. 22])**.**

(a) *If a variety has the Robinson property, then it has the deductive interpolation property.*

(b) *If a variety has the deductive interpolation property and the extension property, then it has the Robinson property.*

2 The weak Robinson property

Theorem 2 naturally poses the challenge of defining a property weaker than the EP that, when combined with the DIP, is equivalent to the RP (and hence also the AP). To address this challenge, we define the weak Robinson property.

We say that a variety $\mathcal{V}$ has the *weak Robinson property* (for short, WRP) if for any $\Sigma \cup \Pi \cup \{\varepsilon\} \subseteq Eq(X)$ such that $Var(\Sigma) \cap Var(\Pi) \neq \emptyset$ and $Var(\{\varepsilon\}) \cap Var(\Pi) \subseteq Var(\Sigma)$, whenever

(i) $\Sigma \models_{\mathcal{V}} \delta \Longleftrightarrow \Pi \models_{\mathcal{V}} \delta$ for all $\delta \in Eq(X)$ with $Var(\delta) \subseteq Var(\Sigma) \cap Var(\Pi)$;

(ii) $\Sigma \cup \Pi \models_{\mathcal{V}} \varepsilon$;

(iii) $\Pi \models_{\mathcal{V}} \rho \implies \Sigma \models_{\mathcal{V}} \rho$ for all $\rho \in Eq(X)$ with $Var(\rho) \subseteq Var(\Sigma)$, then $\Sigma \vDash_{\mathcal{V}} \varepsilon$.

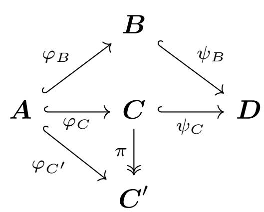
This property provides a positive answer to the challenge posed above. By definition, it is immediate that the RP implies the WRP. Also, it can be shown that the WRP is implied by the EP, and indeed is strictly weaker than the EP, since, e.g., the variety of groups has the RP but not the EP. The following theorem states that the conjunction of the WRP and DIP are in fact equivalent to the RP.

Theorem 3. *A variety has the Robinson property if and only if it has the weak Robinson property and the deductive interpolation property.*

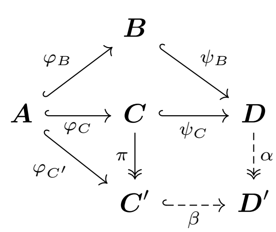
This theorem implies that there are varieties that lack the WRP but still satisfy the DIP—as observed in [7], the variety of semigroups has the DIP despite failing the AP.

In parallel to the connection between the EP and CEP, there exists an algebraic counterpart of the WRP. We say that a variety $\mathcal{V}$ has the *weak congruence extension property* (for short, WCEP) if for any algebra $\boldsymbol{D} \in \mathcal{V}$ with subalgebras $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{V}$ such that $\boldsymbol{A}$ is a common subalgebra of $\boldsymbol{B}$ and $\boldsymbol{C}$, and $\boldsymbol{D}$ with the inclusion maps $\boldsymbol{B} \hookrightarrow \boldsymbol{D}$, $\boldsymbol{C} \hookrightarrow \boldsymbol{D}$ is the pushout of the inclusion maps $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$, $\boldsymbol{A} \hookrightarrow \boldsymbol{C}$, the following holds: for every congruence Θ of $\boldsymbol{C}$ such that $\Theta \cap A^2$ is the least congruence Δ_A of $\boldsymbol{A}$, there exists a congruence Φ of $\boldsymbol{D}$ such that $\Theta = \Phi \cap C^2$.

A categorical approach provides a natural way to characterize properties such as the CEP using diagrams (see, e.g., [1]). Similarly, we can give a categorical description of the WRP. We say that a variety $\mathcal{V}$ has the *weak extension property* (for short, WEP) if for each commuting diagram in $\mathcal{V}$ of the form



where $\varphi_{C'}$ is injective, π is surjective and $\boldsymbol{D}$ with the embeddings $\psi_B \colon \boldsymbol{B} \to \boldsymbol{D}$ and $\psi_C \colon \boldsymbol{C} \to \boldsymbol{D}$ is the pushout of the doubly injective span $\langle \boldsymbol{A}, \varphi_B, \varphi_C \rangle$, there exist a surjective homomorphism $\alpha \colon \boldsymbol{D} \to \boldsymbol{D}'$ and an embedding $\beta \colon \boldsymbol{C}' \to \boldsymbol{D}'$, such that the following diagram commutes:



Theorem 4. *Let $\mathcal{V}$ be a variety. Then the following are equivalent:*

(1) *$\mathcal{V}$ has the weak Robinson property.*

(2) *$\mathcal{V}$ has the weak congruence extension property.*

(3) *$\mathcal{V}$ has the weak extension property.*

3 Concluding remarks

Despite the progress made, there remain several intriguing gaps in our understanding. Crucially, we do not yet have any example of a variety that has the WRP but lacks the AP and CEP. Even if these properties are distinct in the setting of universal algebra, it would be interesting to identify families of varieties that have or do not have the WRP as a method either for refuting the RP or for showing that it is equivalent to the DIP. For example, it is known that the variety of lattice-ordered groups does not have the RP, but the question of whether it has the DIP is open; by showing that this variety has the WRP, the question would be answered negatively.

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