Amalgamation failures in MTL-algebras

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In this contribution we present some new results concerning the deductive interpolation property in substructural logics, via the study of amalgamation in their equivalent algebraic semantics which are classes of residuated lattices. In particular, we solve a longstanding open problem, showing that the following varieties do not have the amalgation property: MTL-algebras and its involutive and pseudocomplemented subvarieties, IMTL and SMTL.

Residuated structures play an important role in the field of algebraic logic; their equivalent algebraic semantics, in the sense of Blok and Pigozzi [1], encompass many of the interesting nonclassical logics: intuitionistic logic, intermediate logics, many-valued logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics.

Let us be more precise; a residuated lattice is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ of type $(2, 2, 2, 2, 2, 0)$ such that: $(A, \vee, \wedge)$ is a lattice; $(A, \cdot, 1)$ is a monoid; the residuation law holds: for all $x, y, z \in A$, $x \cdot y \leq z \Leftrightarrow y \leq x\backslash z \Leftrightarrow x \leq z/y$, (where $\leq$ is the lattice ordering). Residuated lattices form a variety. A residuated lattice is said to be: *integral* if the monoidal identity is the top element of the lattice; *commutative* if the monoidal operation is commutative; n-potent if it holds that $x^n = x^{n+1}$; *semilinear* if it is a subdirect product of chains.

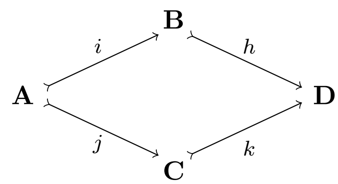
Residuated lattices with an extra constant 0 are called FL-algebras (since they are the equivalent algebraic semantics of the Full Lambek calculus, see [3]), and we call them 0-*bounded* if $0 \leq x$; *bounded* if they are integral and 0-bounded. Semilinear bounded commutative FL-algebras are called MTL-algebras since they are the equivalent algebraic semantics of the monoidal t-norm based logic MTL [2]. Among the most relevant subvarieties of MTL-algebras we have: IMTL-algebras, given by involutive algebras ($\neg\neg x = x$), and SMTL-algebras, i.e. the pseudocomplement subclass ($x \wedge \neg x = 0$).

Our results use one of the most interesting *bridge theorems* that are a consequence of algebraizability: the connection between logical interpolation properties and algebraic amalgamation properties. We say that a logic $\mathcal{L}$, associated to a consequence relation $\vdash$, has the *deductive interpolation property* if for any set of formulas $\Gamma \cup \{\psi\}$, if $\Gamma \vdash \psi$ then there exists a formula δ such that $\Gamma \vdash \delta$, $\delta \vdash \psi$ and the variables appearing in δ belong to the intersection of the variables appearing both in Γ and in ψ, in symbols $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$.

If the logic $\mathcal{L}$ has a variety V as its equivalent algebraic semantics, and V satisfies the *congruence extension property* (CEP), $\mathcal{L}$ has the deductive interpolation property if and only if V has the *amalgamation property* (without the CEP, the amalgamation property corresponds to the stronger Robinson property, see [8]).

Let us then recall the other necessary notions.

Definition 1. *Given a class* K *of algebras in the same signature, a* V-formation *is a tuple* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ *where* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ *and* i, j *are embeddings of* $\mathbf{A}$ *into* $\mathbf{B}$ *and* $\mathbf{C}$ *respectively; an* amalgam *in* K *for the V-formation* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ *is a triple* $(\mathbf{D}, h, k)$ *where* $\mathbf{D} \in \mathsf{K}$ *and* h *and* k *are embeddings of respectively* $\mathbf{B}$ *and* $\mathbf{C}$ *into* $\mathbf{D}$ *such that* $h \circ i = k \circ j$.

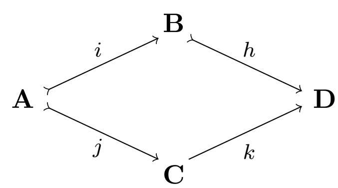


A class K *of algebras has the* amalgamation property *if for any V-formation in* K *there is an amalgam in* K*.*

We focus on the study of the amalgamation property in semilinear varieties of residuated lattices, solving some long-standing open problems; most importantly, we establish that semilinear commutative (integral) residuated lattices and their 0-bounded versions do not have the amalgamation property (i.e., MTL-algebras and their 0-free subreducts).

In order to obtain a failure of the amalgamation property, we use the recent results in [4]; the authors show that in a variety V with the CEP and whose class of finitely subdirectly irreducible members $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras, the amalgamation property of the variety is equivalent to the so-called *one-sided amalgamation property* of $\mathsf{V}_{\mathrm{FSI}}$.

Definition 2. *Given a V-formation* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$*, a* one-sided amalgam *for it is a triple* $(\mathbf{D}, h, k)$ *with* $\mathbf{D} \in \mathsf{K}$ *and as for amalgamation* $h \circ i = k \circ j$*, but while* h *is an embedding,* k *is a homomorphism.*



A class K *of algebras has the one-sided amalgamation property if for any V-formation there is a one-sided amalgam in* K*.*

The mentioned result of [4] is particularly useful in varieties generated by commutative residuated chains; indeed, all commutative residuated lattices have the CEP and a semilinear residuated lattice is finitely subdirectly irreducible if and only if it is totally ordered. Hence, in order to show the failure of the amalgamation property in a semilinear variety with the congruence extension property, it suffices to find a V-formation whose algebras are totally ordered, and that does not have a one-sided amalgam in residuated chains. We do exactly this, and we exhibit a V-formation, which we call $\mathcal{VS}$-formation, given by 2-potent commutative integral residuated chains that does not have a one-sided amalgam in the class of totally ordered residuated lattices. This entails that, if V is a variety of semilinear residuated lattices with the congruence extension property, and such that the algebras in the $\mathcal{VS}$-formation belong to V, then V does not have the amalgamation property. In particular we get the following results.

Theorem 3 ([7])**.** *The following varieties do not have the amalgamation property:*

1. *MTL-algebras;*
2. *Semilinear commutative residuated lattices;*
3. *Semilinear commutative integral residuated lattices;*

4. *Semilinear commutative* FL*-algebras;*

5. *n-potent MTL-algebras for* $n \geq 2$*.*

The result about semilinear commutative residuated lattices has recently been shown in [5]. Using some algebraic constructions (*rotations* and *liftings*) we are also able to adapt our counterexample to construct a V-formation consisting of, respectively, involutive and pseudo-complemented FL-algebras; thus in particular we obtain the following:

Theorem 4 ([7])**.** *The following varieties do not have the amalgamation property:*

1. *IMTL-algebras;*

2. *SMTL-algebras;*

3. *n-potent IMTL and SMTL-algebras for* $n \geq 2$*.*

We observe that given the previously mentioned bridge theorem, our results entail that the logics corresponding to the varieties in Theorems 3 and 4 do not have the deductive interpolation property.

Finally, we mention that the algebras involved in the V-formation that yields the counterexample can be constructed by means of a new construction that we introduce in order to be able to construct new chains from known ones. Such construction extends and generalizes the *partial gluing construction* in [6], and allows us to find other countably many varieties of residuated lattices without the amalgamation property.

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