Axiomatising non falsity and threshold preserving variants of MTL logics

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Fuzzy logics are logics of *graded truth* that have been proposed as a suitable tool for reasoning with imprecise information, in particular for reasoning with propositions containing vague predicates. Their main feature is that they allow to interpret formulas in a linearly ordered scale of truth-values, and this is specially suited for representing the gradual aspects of vagueness. In particular, systems of fuzzy logic have been in-depth developed within the frame of mathematical fuzzy logic [3] (MFL). Most well known and studied systems of mathematical fuzzy logic are the so-called *t-norm based fuzzy logics*, corresponding to formal many-valued calculi with truth-values in the real unit interval $[0,1]$ and with a conjunction and an implication interpreted respectively by a (left-) continuous t-norm and its residuum, and thus, including e.g. the well-known Łukasiewicz and Gödel infinitely-valued logics, corresponding to the calculi defined by Łukasiewicz and min t-norms respectively. The most basic t-norm based fuzzy logic is the logic MTL (monoidal t-norm based logic) introduced in [6]. In logical systems in MFL, the usual notion of deduction is defined by requiring the preservation of the truth-value 1 (full *truth-preservation*), which is understood as representing the absolute truth. For instance, let L be any extension of MTL, which we assume to be complete w.r.t. the family $\mathcal{C}_L = \{[\mathbf{0},\mathbf{1}]_* \mid [\mathbf{0},\mathbf{1}]_* \text{ is a L-algebra}\}$ of standard L-algebras. Then the typical notion of logical consequence is the following for every set of formulas $\Gamma \cup \{\varphi\}$:

$\Gamma \models_L \varphi$ if, for any $[0,1]_* \in \mathcal{C}_L$ and any $[0,1]_*$-evaluation e, if $e(\psi) = 1$ for any $\psi \in \Gamma$, then $e(\varphi) = 1$ as well.

In [2], Bou, Esteva et al. introduced the degree preserving MTL-logics where they change the (full) truth paradigm to the degree preserving paradigm, in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is greater or equal than those of the premises. For any extension L of MTL complete w.r.t. the family $\mathcal{C}_L$ of standard L-algebras the degree preserving variant of L, denoted by $L^{\leqslant}$ is defined as

$\Gamma \models_L^{\leqslant} \varphi$ if, for any $[0,1]_* \in \mathcal{C}_L$, any $[0,1]_*$-evaluation e and for any $a \in [0,1]$, if $e(\psi) \geqslant a$ for any $\psi \in \Gamma$, then $e(\varphi) \geqslant a$.

As a matter of fact, the degree preserving logic $L^{\leqslant}$ is strongly related to the 1-preserving logic L. Indeed, on the one hand, it holds that $\models_L^{\leqslant} \varphi$ iff $\models_L \varphi$, so both logics share the set of valid formulas. Moreover, if for any finite set of formulas Γ we let $\Gamma^{\wedge} = \wedge\{\psi \mid \psi \in \Gamma\}$, we can observe that

$$\Gamma \models_L^{\leqslant} \varphi \text{ iff } \models_L \Gamma^{\wedge} \to \varphi,$$

and hence, iff $\models_L^{\leqslant} \Gamma^{\wedge} \to \varphi$. This property can be seen as a sort of deduction theorem for $\models_L^{\leqslant}$.

It has been shown in [2] that in the case the logic L has a complete axiomatisation with Modus Ponens as the only inference rule, then the logic $Ł^{\leqslant}$ admits a complete axiomatisation as well, having as axioms the axioms of L and as inference rules the rule of adjunction:

$$\text{(Adj)} \quad \frac{\varphi, \quad \psi}{\varphi \wedge \psi},$$

and the following restricted form of the Modus Ponens rule

$$\text{(r-MP)} \quad \frac{\varphi, \quad \varphi \to \psi}{\psi}, \quad \text{if } \vdash_L \varphi \to \psi.$$

If the logic L has additional inference rules

$$(\mathrm{R}_i) \quad \frac{\Gamma_i}{\varphi}$$

for $i \in I$, then [4, Proposition 1] shows that $L^{\leqslant}$ is axiomatised with the above axioms and rules together with the following restricted forms of the rules (R_i):

$$(\text{r-R}_i) \quad \frac{\Gamma_i}{\varphi}, \quad \text{if } \vdash_L \Gamma_i.$$

Still, another way of defining different variants of a fuzzy logic is put forward in [1], although for the particular case of Łukasiewicz fuzzy logic. In this approach, the notion of consequence at work is the *non-falsity preservation*, according to which a conclusion follows from a set of premises whenever if the premises are non-false, so must be the conclusion. In other words, assuming a $[0,1]$-valued semantics, this is the case when, for any evaluation, if truth degrees of the premises are above 0, then the truth-degree of the conclusion is so as well. For any extension L of MTL complete w.r.t. the family $\mathcal{C}_L$ of standard L-algebras we define the following non falsity preserving variant:

$$\Gamma \models_L^{(0} \varphi \quad \text{if,} \quad \begin{array}{l} \text{for any } [0,1]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e\,, \\ \text{if } e(\psi) > 0 \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) > 0. \end{array}$$

The purpose of this talk is to obtain a similar type of axiomatisations for some non-falsity preserving logics. First observe that for any truth preserving logic L with standard semantics and for any formula φ it is obvious that $\models_L \varphi$ implies $\models_L^{(0} \varphi$, so the set of valid formulas of L is contained in the set of valid formulas of the non falsity preserving variant $L^{(0}$. Moreover the finitary versions of both logics are strongly related.

Lemma 1. *For every pair of formulas φ, ψ the following relation holds:*

$$\varphi \models_L^{(0} \psi \quad \textit{iff} \quad \neg\psi \models_L \neg\varphi.$$

We now focus on logics defined by classes of standard IMTL-algebras (standard MTL-algebras with an involutive negation). We remind that this means that $*$ is a left-continuous t-norm such that the residual negation $\neg$, defined as $\neg x = x \to 0 = \sup\{y \in [0,1] \mid x * y = 0\}$ satisfies the involutivity condition $\neg(\neg x) = x$. Notable examples of such t-norms are Łukasiewicz t-norm (which is continuous) and Nilpotent Minimum t-norm.

Assume L is an axiomatic extension of IMTL, complete w.r.t. a class of standard algebras $\mathcal{C}_L$, and whose corresponding notion of proof is denoted $\vdash_L$. It is immediate to observe that in the case of a IMTL logic L, Lemma 1 can be strengthened in the sense that the 1-preserving logic L and the non-falsity preserving logic $L^{(0}$ become interdefinable. Namely,

(i) $\varphi \models_L \psi$ iff $\neg\psi \models_L^{(0} \neg\varphi$, (ii) $\varphi \models_L^{(0} \psi$ iff $\neg\psi \models_L \neg\varphi$.

In order to syntactically characterise $\models_L^{(0}$, the following system nf-L, called the *non-falsity preserving companion* of L, is defined in [5] as follows.

Definition 1. *The calculus* nf-L *is defined by the following axioms and rules:*

- *Axioms of L*
- *Rule of Adjunction: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Reverse Modus Ponens: (MP^r)* $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee \ \neg(\varphi \to \psi) \ \vee \ \chi}$
- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$, *if* $\vdash_L \varphi \to \psi$

The above (MP^r) rule captures the following form of reverse of modus ponens: if $\neg\psi$ is non-false then either $\neg\varphi$ is non-false or $\neg(\varphi \to \psi)$ is non-false. The addition of the disjunct χ both in the premise and in the conclusion of the rule is needed for technical reasons.

The following is a syntactic counterpart of part of Lemma 1.

Proposition 2. *If* $\psi \vdash_L \varphi$ *then* $\neg\varphi \vdash_{\text{nf-}L} \neg\psi$.

Thanks to this relation, the logic nf-L has been shown to be complete with respect to the intended semantics.

Theorem 3. *Let L be an axiomatic extension of IMTL. Then, the calculus* nf-*L is sound and complete w.r.t. the finitary logic of* $L^{(0}$.

Note that, as a direct corollary, Definition 1 provides us with complete axiomatisations of non-falsity preserving companions of prominent IMTL logics like Łukasiewicz logic or Nilpotent Minimum logic.

We are also able to prove similar result as in the previous theorem without the requirement of the negation $\neg$ to be involutive. Indeed, let $\text{MTL}_{\neg\neg}$ be the (non-axiomatic) extension of MTL with the rule

$$(\text{R}_{\neg\neg}) \quad \frac{\neg\neg\varphi}{\varphi}.$$

The algebraic semantics of $\text{MTL}_{\neg\neg}$ consists of the quasi-variety generated by the class of MTL-chains **A** such that its negation $\neg$ is such that, for any $a \in A$, $\neg a = 0$ iff $a = 1$, or equivalently $\neg a > 0$ iff $a < 1$. If L is an axiomatic extension of MTL, let us denote by $\text{L}_{\neg\neg}$ the extension of L with the rule $(\text{R}_{\neg\neg})$. If L is complete w.r.t. a class of standard algebras $\mathcal{C}_L$, then $L_{\neg\neg}$ is also complete w.r.t. the class of standard algebras $\mathcal{C}_{L_{\neg\neg}}$. Moreover, in $L_{\neg\neg}$ we keep having at the semantical level the equivalence between the 1-preserving logic and the non-falsity preserving logic, in the following sense.

Lemma 4. *For any fuzzy logic L, then the following conditions hold:*

(i) $\varphi \models_{L_{\neg\neg}} \psi$ *iff* $\neg\psi \models_{L_{\neg\neg}}^{(0} \neg\varphi$, *(ii)* $\varphi \models_{L_{\neg\neg}}^{(0} \psi$ *iff* $\neg\psi \models_{L_{\neg\neg}} \neg\varphi$.

Then one can define the non-falsity preserving companion of a $\mathrm{MTL}^{\neg\neg}$-logic and prove its completeness as follows. In fact, we can restrict ourselves to extensions of MTL logics with the rule $(\mathrm{R}_{\neg\neg})$. where $\neg(\varphi \wedge \neg\varphi)$ is not a tautology, that is extensions of non SMTL-logics with the rule $(\mathrm{R}_{\neg\neg})$. Indeed, note that if L is an SMTL logic, then $L_{\neg\neg}$ collapses into classical logic.

Theorem 5. *Let L be an axiomatic extension of MTL that is non-SMTL. Then the calculus* $\mathsf{nf}\text{-}L_{\neg\neg}$, *defined by the following axioms and rules:*

- *Axioms of L*
- *The rule* $(R_{\neg\neg})$
- *The rule of adjunction (Adj)*
- *The rule of Reverse Modus Ponens* (MP^r)
- *The rule of Restricted Modus Ponens (r-MP)*

is a sound and complete axiomatisation w.r.t. to the finitary logic of $L^{(0}_{\neg\neg}$.

Finally, we turn our attention to logics preserving lower bounds of truth-values. Let L be an extension (or expansion) of MTL complete w.r.t. some class of standard L-algebras $\mathcal{C}_L$, fix some positive value $a \in (0,1]$, we define the logic L^a as follows:

$$\Gamma \models^a_L \varphi \quad \text{if,} \quad \begin{array}{l} \text{for any } [0,1]_* \in \mathcal{C}_L \text{, any } [0,1]_*\text{-evaluation } e, \\ \text{if } e(\psi) \geqslant a \text{ for any } \psi \in \Gamma \text{, then } e(\varphi) \geqslant a. \end{array}$$

We will end the talk by discussing some general but sufficient assumptions on L to guarantee a finitary axiomatisation of L^a.

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