Glueings that produce commutative idempotent involutive residuated lattices

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Extended abstract

In [3], it was shown that every locally integral involutive partially ordered semigroup (ipo-semigroup) $\mathbf{A} = (A, \leqslant, \cdot, \sim, -)$, and in particular every locally integral involutive semiring, decomposes in a unique way into a family $\{\mathbf{A}_p : p \in A^+\}$ of integral ipo-monoids, which are called its *integral components*. In the semiring case, the integral components are actually unital semirings. Moreover, it was shown that there is a family of monoid homomorphisms $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q\}$, indexed over the positive cone $(A^+, \leqslant)$, such that the structure of $\mathbf{A}$ can be recovered as a glueing $\int_\Phi \mathbf{A}_p$ of its integral components along Φ. Reciprocally, necessary and sufficient conditions were given such that the Płonka sum of any family of integral ipo-monoids $\{\mathbf{A}_p : p \in D\}$, indexed over a join-semilattice $(D, \vee)$ along a family of monoid homomorphisms Φ is an ipo-semigroup. Further generalizations were obtained in [1] for a larger class of partially ordered structures, not necessarily integral. But, it's important to notice that, in general, the obtained sums fail to be lattice-ordered, even if this is the case for all of its components.

The results of the works above are generalization of [4], in which a step-by-step construction for two finite commutative idempotent involutive residuated lattices was shown to produce lattice-ordered algebras, rather than just partially ordered algebras. This glueing is realized via an isomorphism between an ideal and a filter (jointly called an *interface*) of the corresponding algebras, and the join and meet of the glueing are explicitly defined using this isomorphism. But, this kind of *interface glueings* lack the flexibility of the Płonka sums, since only two algebras can be glued at a time.

In this work we analyze the two different approaches (Płonka sums and interface glueing) and find characterizations for small configurations of integral involutive residuated lattice components to be lattice-ordered. In the setting of involutive residuated lattices without finiteness, commutativity or idempotence, we show that the interface glueing of two integral components again produces an involutive residuated lattice. In the case of Boolean components, slightly larger configurations can be characterized.

More generally, if $\mathbf{B} = \int_\Phi \mathbf{B}_p$ is a commutative idempotent involutive residuated lattice whose positive elements B^+ form a distributive lattice, and we also assume that each integral component of $\mathbf{B}$ is complete and atomic, then we conjecture that $\mathbf{B}$ is lattice-ordered if and only if

1. each $\varphi_{pq} \in \Phi$ restricts to a Boolean isomorphism between a principal ideal of $\mathbf{B}_p$ and the filter generated by $\varphi_{pq}(0_p)$ in $\mathbf{B}_q$ and,

2. for each $q, r \in B^+$, if q and r are incomparable with $p = q \wedge r$ and $s = qr$

   - $\varphi_{ps}(0_p) = \varphi_{qs}(0_q) \vee^s \varphi_{rs}(0_r)$,
   - $\varphi_{pq}(0_p) \leqslant \varphi'_{qs}(s)$ (and the same for r), and
   - $\varphi'_{ps}(s) = \varphi'_{pq}(q) \cdot \varphi'_{pr}(r)$.

References

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