Decidability of Bernays–Schönfinkel Class of Gödel Logics *

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It is widely acknowledged that any first-order formula in classical logic is logically identical to one in prenex form. In general, any set of quantifier prefixes defines a fragment of first-order logic, specifically the set of prenex formulas that contain one of the quantifier prefixes in question. In the early stages of research, it was recognised that while some fragments defined in this way have decidable satisfiability/validity, others do not.

In 1928, P. Bernays and M. Schönfinkel proved the decidability for the class of function-free sentences with prefixes $\exists\bar{x}\forall\bar{y}A(\bar{x},\bar{y})$ (satisfiability) and $\forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ (validity) (specifically, the set of sentences that, when written in prenex normal form, have a prefix containing quantifiers and the matrix without function symbols) [5]. We will study the decidability of the Bernays–Schönfinkel class for all Gödel logics. Our argument for validity is based on the fact that Skolemization is possible for prenex Gödel logics and our argument for satisfiability is based on the general properties of prenex formulas. We must note that in Gödel logics validity and satisfiability are not dual as in classic logic.

Definition 1. *(Gödel logics). First-order Gödel logics are a family of many-valued logics where the truth values set (known also as* Gödel set*) V is closed subset of the full $[0,1]$ interval that includes both 0 and 1 given by the following evaluation function $\mathcal{I}$ on V*

$$
\begin{aligned}
&(1) \quad \mathcal{I}(\bot) = 0\\
&(2) \quad \mathcal{I}(A \wedge B) = min\{\mathcal{I}(A), \mathcal{I}(B)\}\\
&(3) \quad \mathcal{I}(A \vee B) = max\{\mathcal{I}(A), \mathcal{I}(B)\}\\
&(4) \quad \mathcal{I}(A \supset B) = \begin{cases} \mathcal{I}(B) & if\ \mathcal{I}(A) > \mathcal{I}(B), \\ 1 & if\ \mathcal{I}(A) \leq \mathcal{I}(B). \end{cases}\\
&(5) \quad \mathcal{I}(\forall x A(x)) = inf\{\mathcal{I}(A(u))\, u \in U_{\mathcal{I}}\}\\
&(6) \quad \mathcal{I}(\exists x A(x)) = sup\{\mathcal{I}(A(u))\, u \in U_{\mathcal{I}}\}
\end{aligned}
$$

Definition 2. *(1-entailment). For a truth value set V, a (possibly infinite) set Γ of formulas (1-)entails a formula A if the interpretation $\mathcal{I}$ on V of A is 1 in case the interpretations of all formulas in Γ are 1, i.e., $\Gamma \Vdash_V A \Longleftrightarrow (\forall \mathcal{I}, \forall B \in \Gamma : \mathcal{I}(B) = 1) \to \mathcal{I}(A) = 1$.*

As a generalization of classical satisfiability, we introduce the following concepts:

Definition 3 (Validity)**.** *The formula in Gödel logic is* valid *if the formula evaluates to 1 under every interpretation.*

Definition 4 (satisfiability)**.** *The formula in Gödel logic is* 1-satisfiable *if there exists at least one interpretation that assigns 1 to the formula.*

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Validity in Berneys-Schönfinkel class for all Gödel logics is Decidable

Definition 5 (Structural Skolem form). *Let A be a closed first-order formula. Whenever A does not contain strong quantifiers, we define its* structural Skolem form *as $A^S = A$.*
Suppose now that A contains strong quantifiers. Let (Qy) be the first strong quantifier occurring in A. If (Qy) is not in the scope of weak quantifiers, then its structural Skolem form is

$$A^S = (A_{-(Qy)}\{y \leftarrow c\})^S,$$

where $A_{-(Qy)}$ is the formula A after omission of (Qy) and c is a constant symbol not occurring in A. If (Qy) is in the scope of the weak quantifiers $(Q_1x_1)\dots(Q_nx_n)$, then its structural Skolemization is

$$A^S = (A_{-(Qy)}\{y \leftarrow f(x_1,\dots,x_n)\})^S,$$

where f is a function symbol (Skolem function) and does not occur in A.

In Gödel logics, valid prenex formulas can be sharpened to validity equivalent purely existential formulas by Skolemization.

Lemma 1. *(Skolemization) For all prenex formulas $Q\bar{x}A(\bar{x})$ and all Gödel logics G*

$$\Gamma \Vdash_G Q\bar{x}A(\bar{x}) \Longleftrightarrow \Gamma \Vdash_G (Q\bar{x}A(\bar{x}))^S$$

where $Q\bar{x}$ is a quantifier prefix and $A(\bar{x})$ is a quantifier-free formula.

Proof. It is sufficient to prove with A arbitrary and f a new function:

$$\Gamma \Vdash_G \exists\overline{x}\forall y A(\overline{x}, y) \Leftrightarrow \Gamma \Vdash_G \exists\overline{x}A(\overline{x}, f(\overline{x})).$$

It follows then from induction. ($\Rightarrow$) The direction from left to right is obvious.
($\Leftarrow$) For the other direction, if $\nVdash_G \exists\overline{x}\forall y A(\overline{x}, y)$ then for some interpretation $\mathcal{I}$

$$\sup\{d_{\overline{c}} \mid \mathcal{I}(\forall y A(\overline{c}, y)) = d_{\overline{c}}\} \le d < 1.$$

Using the axiom of choice we can assign a value for every $f(\overline{c})$ such that $\mathcal{I}(A(\overline{c}, f(\overline{c}))$ is in between $d_{\overline{c}}$ and $d_{\overline{c}} + \frac{1-d}{2}$. As a consequence

$$\sup\{d_{\overline{c}} + \frac{1-d}{2} \mid \mathcal{I}(A(\overline{c}, f(\overline{c}))) \le d_{\overline{c}} + \frac{1-d}{2}\} \le d + \frac{1-d}{2} < 1$$

and thus $\Gamma \nVdash_G \exists\overline{x}A(\overline{x}, f(\overline{x}))$. □

Theorem 1. *Validity in Berneys-Schönfinkel (BS) class is decidable for all Gödel logics.*

Proof. from above lemma follows

$$\Gamma \Vdash_G \forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y}) \Longleftrightarrow \Gamma \Vdash_G \exists\bar{y}A(\bar{c},\bar{y})$$

for new constants $\bar{c}$. Suppose there is a countermodel M such that $M \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$. Then there is also a countermodel M' such that $M' \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$ where the domain of M' contains only interpretations of $\bar{c}$. □

Corollary 1. *1) Let $\exists\bar{y}A(\bar{y})$ contain only constants $\bar{c}$, then Herbrand's theorem holds for $\exists\bar{y}A(\bar{y})$ for all Gödel logics G.*

2) Let $\forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ prenex formulas contain only constants $\bar{d}$, then $\Gamma \Vdash_G \forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y}) \iff \Gamma \Vdash_{G'} \forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ for all infinitely-valued Gödel logics G, G'.

Proof. 1) According to the proof of the above theorem, $M \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$ implies $M' \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$ with restricted domain to constants only.
2) follows from 1), as Herbrand disjunction is contained in $\bigvee_n A(\bar{c}_n,\bar{d}_n)$ where $\bar{c}_n,\bar{d}_n$ are possible variations of $\bar{c},\bar{d}$ and validity for propositional formulas coincides with infinitely-valued Gödel logics. □

Remark 1. *Note that 1) is not trivial as prenex formulas and consequently $\exists$-formulas (see. Lemma 1) for countable Gödel logics are not r.e. [4].*

1-satisfibility in Berneys-Schönfinkel class for all Gödel logics is Decidable

Lemma 2 (Gluing lemma)**.** *Let $\mathcal{I}$ be an interpretation into $V \subseteq [0,1]$. Let us fix a value $\omega \in [0,1]$ and define*

$$\mathcal{I}_\omega(\mathcal{P}) = \begin{cases} \mathcal{I}(\mathcal{P}) & \text{if } \mathcal{I}(\mathcal{P}) \leq \omega, \\ 1 & \text{otherwise} \end{cases}$$

for atomic formula $\mathcal{P}$ in $\mathcal{L}^{\mathcal{I}}$. Then $\mathcal{I}_\omega$ is an interpretation into V such that

$$\mathcal{I}_\omega(\mathcal{B}) = \begin{cases} \mathcal{I}(\mathcal{B}) & \text{if } \mathcal{I}(\mathcal{B}) \leq \omega, \\ 1 & \text{otherwise} \end{cases}$$

As an immediate consequence, we have:

Corollary 2. *Prenex formulas in Gödel logics admit 1-satisfiability iff they are classical saitisfiable.*

Theorem 2. *1-satisfiability in Berneys-Schönfinkel class is decidable for all Gödel logics.*

Proof. The proof is obvious as 1-satisfiability coincides with classical satisfiability and, therefore, is decidable. □

Remark 2. *All Gödel logics coincide for the Bernays-Schönfinkel class w.r.t. 1-satisfiability, but only the infinitely valued Gödel logics coincide for the Bernays-Schönfinkel class w.r.t. to validity. The Berneys-Schönfinkel fragment of any infinitely-valued Gödel logic is the intersection of the Bernays-Schönfinkel fragments of the finitely-valued Gödel logic, both for satisfiability and validity.*

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