The structure of the ℓ-pregroup $\mathbf{F}_n(\mathbf{Z})$

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Abstract

Lattice-ordered pregroups (ℓ-pregroups) are exactly the involutive residuated lattices where addition and multiplication coincide. Among them, for every n, the n-periodic ℓ-pregroup $\mathbf{F}_n(\mathbb{Z})$ of n-periodic order-preserving functions on $\mathbb{Z}$ plays an important role in understanding distributive ℓ-pregroups and also n-periodic ones. We study the structure of this algebra in great detail and provide order-theoretic and monoidal-theoretic descriptions. This then paves the way for axiomatizing the variety generated by $\mathbf{F}_n(\mathbb{Z})$, covered in a different submission.

1 Introduction

A *lattice-ordered pregroup* (*ℓ-pregroup*) is an algebra $(A, \wedge, \vee, \cdot, ^\ell, ^r, 1)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leq$, and for all $x \in A$,

$$x^\ell x \leq 1 \leq xx^\ell \text{ and } xx^r \leq 1 \leq x^r x.$$

We often refer to x^ℓ and x^r as the *left* and *right inverse* of x, respectively. The well-studied lattice-ordered groups (ℓ-groups) are exactly the ℓ-pregroups where the two inverses coincide: $x^\ell = x^r$. Also, ℓ-pregroups constitute lattice-ordered versions of *pregroups*, which are ordered structures introduced by Lambek [11] in the study of applied linguistics, where they are used to describe sentence patterns in many natural languages; they have also been studied extensively by Buzkowski [1] and others in the context of mathematical linguistics in connection to context-free grammars. Pregroups where the order is discrete (and also pregroups that satisfy $x^\ell = x^r$) are exactly groups.

The main reason for our interest in ℓ-pregroups is that they are precisely the *involutive residuated lattices* that satisfy $x + y = xy$; in that respect their study is connected to the algebraic semantics of *substructural logics* [8].

It is easy to show that the underlying lattices of ℓ-groups are distributive, but it remains an open problem whether every ℓ-pregroup is distributive. Partial answers to this question include [7], where it is shown that ℓ-pregroups are semidistributive, and [5], where it is shown that all *periodic* (see below) ℓ-pregroups are distributive. We denote by DLP the variety of *distributive ℓ-pregroups*.

In analogy to Cayley's theorem for groups, Holland's *embedding theorem* [9] shows that every ℓ-group can be embedded into a symmetric ℓ-group $\mathbf{Aut}(\mathbf{\Omega})$—the group of order-preserving permutations on a totally ordered set $\mathbf{\Omega}$. Also, Holland's *generation theorem* [10] states that $\mathbf{Aut}(\mathbb{Q})$ generates the variety of ℓ-groups and this is further used to show that the equational theory of ℓ-groups is decidable. In [4] it is shown that every distributive ℓ-pregroup embeds into a functional ℓ-pregroup $\mathbf{F}(\mathbf{\Omega})$ (a generalization of a symmetric ℓ-group), where $\mathbf{\Omega}$ is a chain; actually $\mathbf{\Omega}$ can be taken to be an ordinal sum of copies of the integers, as shown in [2]. Under the general definition where $\mathbf{\Omega}$ is an arbitrary chain, the algebra $\mathbf{F}(\mathbf{\Omega})$ consists of all functions on $\mathbf{\Omega}$ that have residuals and dual residuals of all orders, but in the special case where $\mathbf{\Omega}$ is the chain of the integers, $\mathbf{F}(\mathbb{Z})$ ends up consisting of all order-preserving functions on $\mathbb{Z}$ that are finite-to-one (the preimage of every singleton is a finite set/interval). This representation theorem for distributive ℓ-pregroups is used in [2] to prove an analogue of Holland's generation theorem: the ℓ-pregroup $\mathbf{F}(\mathbb{Z})$ generates the variety DLP (and that its equational theory is decidable).

For every positive integer n, the functions f in $\mathbf{F}(\mathbb{Z})$ that are periodic and have period n end up being exactly the ones that satisfy $f^{\ell^n} = f^{r^n}$ and they form a subalgebra of $\mathbf{F}(\mathbb{Z})$, which we denote by $\mathbf{F}_n(\mathbb{Z})$; here $f^{\ell^3} = f^{\ell\ell\ell}$, for example. In [3] it is proved that DLP is equal to the join of the varieties $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$. This demonstrates the importance of the varieties $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, and hence also the algebras $\mathbf{F}_n(\mathbb{Z})$, in understanding distributive ℓ-pregroups. For example, if an equation fails in DLP, it fails in some $\mathbf{F}_n(\mathbb{Z})$ (and [3] further provides a concrete suitable n).

More generally, in an arbitrary ℓ-pregroup an element x is called *n-periodic* if $x^{\ell^n} = x^{r^n}$; an ℓ-pregroup is called *n-periodic* if all of its elements are, and the corresponding variety is denoted by LP_n. As mentioned before, in [5] it is shown that $\mathsf{LP}_n \subseteq \mathsf{DLP}$, for all n, and in [3] it is further proved that the join of all of the LP_n's is exactly DLP. Thus $\mathsf{DLP} = \bigvee \mathsf{LP_n} = \bigvee \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$. These two appoximations of DLP are quite different since, as shown in [3], the variety $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ is properly contained in $\mathsf{LP_n}$ for every single n. Even though $\mathsf{LP_n} \neq \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, for every n, $\mathbf{F}_n(\mathbb{Z})$ actually plays an important role in understanding LP_n, as well: it is shown in [3] that every n-periodic ℓ-pregroup can be embedded in a *wreath product* of an ℓ-group and $\mathbf{F}_n(\mathbb{Z})$.

2 The structure of the algebra

In [3], enough aspects of $\mathbf{F}_n(\mathbb{Z})$ are studied in order to obtain the above results and also the decidability of the equational theory of $\mathbf{F}_n(\mathbb{Z})$, for all n. However, the lattice-theoretic and monoidal-theoretic structure of $\mathbf{F}_n(\mathbb{Z})$ has been described only for $n = 2$, in [5]. In this contribution we provide a detailed description of $\mathbf{F}_n(\mathbb{Z})$, for all n.

Toward describing the monoidal structure of $\mathbf{F}_n(\mathbb{Z})$ we first identify two of its submonoids: $\boldsymbol{b}^{\mathbb{Z}}$ and $\mathbf{End}(\mathbf{n})$. We denote by $\mathbf{b}$ the function $x \mapsto x+1$ on $\mathbb{Z}$ and by $\boldsymbol{b}^{\mathbb{Z}} := \{\mathbf{b}^k : k \in \mathbb{Z}\}$ the subgroup that it generates; $\boldsymbol{b}^{\mathbb{Z}}$ is an ℓ-group (isomorphic to $\mathbb{Z}$) and it is the largest subgroup/ℓ-subgroup of $\mathbf{F}_n(\mathbb{Z})$ (i.e., the set of all of invertible elements of $\mathbf{F}_n(\mathbb{Z})$). For each n the maps

$$\sigma_n(x) = x \wedge x^{\ell\ell} \wedge \cdots \wedge x^{\ell^{2n-2}} \text{ and } \gamma_n(x) = x \vee x^{\ell\ell} \vee \cdots \vee x^{\ell^{2n-2}},$$

are an interior and a closure operator on $\mathbf{F}_n(\mathbb{Z})$, respectively; also they both have image equal to $\boldsymbol{b}^{\mathbb{Z}}$. In particular, $\mathbf{F}_n(\mathbb{Z})$ is the convex closure of $\boldsymbol{b}^{\mathbb{Z}}$.

For every $n \in \mathbb{Z}$, we denote by $\mathbf{n}$ the n-element chain $0 < 1 < \ldots < n-1$ and by $End(\mathbf{n})$ the endomorphisms (i.e., order-preserving maps) on $\mathbf{n}$. $End(\mathbf{n})$ forms a distributive lattice by pointwise order and a monoid under functional composition, and multiplication distributes over both join and meet; the resulting *distributive lattice-ordered monoid* (DLM) is denoted by $\mathbf{End}(\mathbf{n})$.

The n-periodic extensions to $\mathbb{Z}$ of the functions in $\mathbf{End}(\mathbf{n})$ form a subDLM of $\mathbf{F}_n(\mathbb{Z})$, which we denote by $\mathbf{End}(\mathbf{n})$ as well, by abusing notation. We observe that actually this is only one of the n-many different (overlapping) subDLM of $\mathbf{F}_n(\mathbb{Z})$ that are isomorphic to $\mathbf{End}(\mathbf{n})$. We prove that the union of these copies of $\mathbf{End}(\mathbf{n})$ in $\mathbf{F}_n(\mathbb{Z})$ is equal to the set of flat elements of $\mathbf{F}_n(\mathbb{Z})$, and is contained in the interval $[\mathbf{b}^{1-n}, \mathbf{b}^{n-1}]$; an element x of an ℓ-pregroup is called *flat* if there exist idempotents y, z such that $y \leq x \leq z$.

We prove that every element of $\mathbf{F}_n(\mathbb{Z})$ can be written as a product of the form xy and of the form $y'x'$, where $x, x' \in End(\mathbf{n})$ and $y, y' \in \mathbf{b}^{\mathbb{Z}}$; thus, $F_n(\mathbb{Z}) = End(\mathbf{n}) \cdot \mathbf{b}^{\mathbb{Z}} = \mathbf{b}^{\mathbb{Z}} \cdot End(\mathbf{n})$. Furthermore, we prove that each of these decompositions is unique. However, $\mathbf{F}_n(\mathbb{Z})$ is not isomorphic to the direct product of $\mathbf{End}(\mathbf{n})$ and $\mathbb{Z}$, nor even to a semidirect product of them.

Let $\mathbf{M}$ and $\mathbf{N}$ be monoids, where

- $*\colon M \times N \to N$ a left action of $\mathbf{M}$ on $\mathbf{N}$: $1 * n = n$ and $m_1 * (m_2 * n) = m_1 m_2 * n$,
- $\star\colon M \times N \to M$ a right action of $\mathbf{N}$ in $\mathbf{M}$: $m \star 1 = m$ and $(m \star n_1) \star n_2 = m \star n_1 n_2$,
- $m * n_1 n_2 = (m * n_1)((m \star n_1) * m_2)$ and
- $m_1 m_2 \star n = (m_1 \star (m_2 * n))(m_2 \star n)$.

Then $\mathbf{N} \times_\star^* \mathbf{M} = \langle N \times M, \circ, \langle 1, 1 \rangle \rangle$, where $\langle n_1, m_1 \rangle \circ \langle n_2, m_2 \rangle = \langle n_1(m_1 * n_2), (m_1 \star n_2) m_2 \rangle$, is a monoid called the Zappa product of $\mathbf{N}$ and $\mathbf{M}$ with respect to the two actions. Note that if $\star$ is trivial ($m \star n = m$, for all n, m) or $*$ is trivial, then the Zappa product is a (left or right) semidirect product.

Theorem 1. *The monoid reduct of $\mathbf{F}_n(\mathbb{Z})$ is isomorphic to the Zappa product $\mathbf{End}(\mathbf{n}) \times_\star^* \mathbb{Z}$, where $b^m \star a = c$ and $a * b^m = b^k$, and $c \in \mathbf{End}(\mathbf{n})$ and $k \in \mathbb{Z}$ are the unique elements such that $b^m a = c b^k$.*

Now, to describe the order structure of $\mathbf{F}_n(\mathbb{Z})$ we describe its poset of join irreducibles. We define the poset $\mathbf{C}_n^{\mathbb{Z}} := ([0, n-1] \times \mathbb{Z}, \leq)$ by: for $(m, k), (m', k') \in [0, n-1] \times \mathbb{Z}$,

$$(m, k) \leq (m', k') :\iff m -_n m' \leq (k' - m') - (k - m).$$

As usual for $m, m' \in [0, n-1]$, $m -_n m'$ (difference modulo n) is equal to $m - m'$ if $m \geq m'$ and to $m - m' + n$ if $m < m'$. In particular, $m -_n m' \in [0, n-1]$. Since $0 \leq m -_n m'$, this definition implies $k - m \leq k' - m'$. Also, we note that the corresponding covering relation is

$$(m, k) \prec (m', k') :\iff m + 1 = m' \text{ or } (m = m' \text{ and } k = k' +_n 1).$$

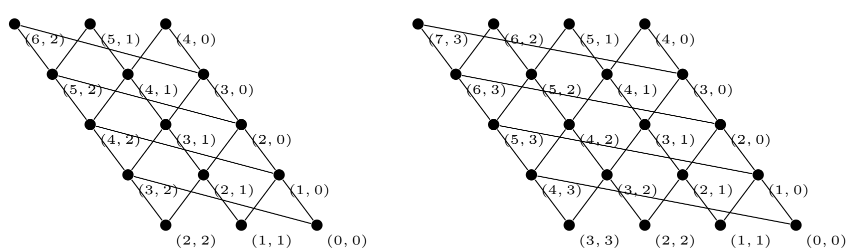

Figure 1: The infinite-layered posets $\mathbf{C}_3^{\mathbb{Z}}$ and $\mathbf{C}_4^{\mathbb{Z}}$.

Theorem 2. *An element of $\mathbf{F}_n(\mathbb{Z})$ is join irreducible iff it is meet irreducible. Also, the poset of join irreducibles of $\mathbf{F}_n(\mathbb{Z})$ is isomorphic to $\mathbf{C}_n^{\mathbb{Z}}$.*

We further define a multiplication on $\mathbf{C}_n^{\mathbb{Z}}$ by:

$$(m', k') \cdot (m, k) = (m, S_n(k - m') + k').$$

Here, for every $n, a \in \mathbb{Z}$, we define $S_n(a) := qn$, where $a = qn + r$, $0 \leq r < n$ and $q, r \in \mathbb{Z}$ are given by the division algorithm; i.e., $S_n(a)$ is the largest whole multiple of n below (and including) a. Note that $(\mathbf{C}_n^{\mathbb{Z}}, \cdot)$ is a semigroup isomorphic to the semidirect product of $(\mathbb{Z}, +)$ and the right-zero semigroup on $[0, n-1]$, where the action $* : [0, n-1] \times \mathbb{Z} \to \mathbb{Z}$ is defined by $m * k := S_n(k - m)$.

Theorem 3. *The join irreducibles of $\mathbf{F}_n(\mathbb{Z})$ form a partially-ordered semigroup that is isomorphic to $(\mathbf{C}_n^{\mathbb{Z}}, \leq, \cdot)$. Also, the join irreducibles of $\mathbf{F}_n(\mathbb{Z})$ are closed under the inverses, and the corresponding operations on $\mathbf{C}_n^{\mathbb{Z}}$ are:*

$$(m,k)^\ell := (k-n+1-S_n(k-n+1), m-S_n(k-n+1)) \text{ and } (m,k)^r := (k-S_n(k), m+n-1-S_n(k)).$$

In view of this result the elements of $\mathbf{F}_n(\mathbb{Z})$ can be viewed as downsets of $\mathbf{C}_n^{\mathbb{Z}}$; then the lattice operations are simply union and intersection, while multiplication and the inversions is simply the element-wise lifting of the multiplication and inversions on $\mathbf{C}_n^{\mathbb{Z}}$.

In view of Theorem 1 we also provide an analysis of $\mathbf{End}(\mathbf{n})$ as a DLM, and of the positive cones of $\mathbf{End}(\mathbf{n})$ and $\mathbf{F}_n(\mathbb{Z})$, via their poset of join irreducibles (as a finitary or one-sided versions of $\mathbf{C}_n^{\mathbb{Z}}$), as well as their multiplicative structure in terms of irreducible generators.

The above analysis can be used to prove the following generation result. The *periodicity* of an element is defined to be the smallest positive k such that the element is k-periodic.

Theorem 4. *$\mathbf{F}_n(\mathbb{Z})$ is generated by any one of its elements of periodicity n.*

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