Varieties of MV-monoids and positive MV-algebras

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Abstract

We investigate MV-monoids and their subvarieties. An *MV-monoid* is an algebra $\langle A, \vee, \wedge, \oplus, \odot, 0, 1 \rangle$ where:

- $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice;

- $\langle A, \oplus, 0 \rangle$ and $\langle A, \odot, 1 \rangle$ are commutative monoids;

- $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;

- for every $x, y, z \in A$,

$$(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z);$$
$$(x \odot y) \oplus ((x \oplus y) \odot z) = (x \oplus (y \odot z)) \odot (y \oplus z);$$
$$(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z;$$
$$(x \oplus y) \odot z = ((x \odot y) \oplus ((x \oplus y) \odot z)) \wedge z.$$

Every MV-algebra in the signature $\{\oplus, \neg, 0\}$ is term equivalent to an algebra that has an MV-monoid as a reduct, by defining, as standard, $1 \coloneqq \neg 0$, $x \odot y \coloneqq \neg(\neg x \oplus \neg y)$, $x \vee y \coloneqq (x \odot \neg y) \oplus y$ and $x \wedge y \coloneqq \neg(\neg x \vee \neg y)$. We study subdirectly irreducible MV-monoids and show that every subdirectly irreducible MV-monoids $\mathbf{A}$ is totally ordered and satisfies property: for all $x, y \in A$, $x \oplus y = 1$ or $x \odot y = 0$.

Furthermore, we investigate the bottom part of the lattice of subvarieties of MV-monoids, characterizing all the almost minimal varieties of MV-monoids as the varieties generated by:

- a reduct of a finite MV-chain of prime order ($\mathbf{Ł}_p^+$);

- the unique MV-monoid $\mathbf{C}_2^\Delta$ on the 3-element chain $0 < \varepsilon < 1$ satisfying $\varepsilon \oplus \varepsilon = \varepsilon$ and $\varepsilon \odot \varepsilon = 0$;

- the dual of $\mathbf{C}_2^\Delta$.

One of the main tool we used to develop the theory of MV-monoids is the categorical equivalence Γ between unit commutative ℓ-monoids and MV-monoids [1].

A *unital commutative ℓ-monoid* is an algebra $\langle M, \vee, \wedge, +, 1, 0, -1 \rangle$ with the following properties:

- $\langle M, \vee, \wedge, +, 0 \rangle$ is a commutative ℓ-monoid;

- $-1 + 1 = 0$;

- $-1 \leq 0 \leq 1$;

- for all $x \in M$ there is $n \in \mathbb{N}$ such that

$$\underbrace{(-1) + \cdots + (-1)}_{n \text{ times}} \leq x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Thus, the relationship between unital commutative ℓ-monoids and MV-monoids is similar to the one between abelian ℓ-groups and MV-algebras and we exploit this fact in several statements of our work.

We also present two versions of Hölder's theorem for unital commutative ℓ-monoids.

Theorem. Let $\mathbf{M}$ be a nontrivial totally ordered unital commutative ℓ-monoid. There is a unique homomorphism from $\mathbf{M}$ to $\mathbb{R}$.

Definition. A unital commutative ℓ-monoid $\mathbf{M}$ is *Archimedean* provided that, for all $x, y \in M$, if for all $n \in \mathbb{N}$ we have $nx \leq ny + 1$, then $x \leq y$.

Theorem. (Hölder's theorem for unital commutative ℓ-monoids) Let $\mathbf{M}$ be an Archimedean nontrivial totally ordered unital commutative ℓ-monoid. The unique homomorphism from $\mathbf{M}$ to $\mathbb{R}$ is injective, and so $\mathbf{M}$ is isomorphic to a subalgebra of $\mathbb{R}$.

Particular examples of MV-monoids are positive MV-algebras, i.e. the $\{\vee, \wedge, \oplus, \odot, 0, 1\}$-subreducts of MV-algebras or, equivalently, the proper subquasivariety of the variety of MV-monoids (MVM), axiomatized relatively to MVM by

$$(x \oplus z \approx y \oplus z \text{ and } x \odot z \approx y \odot z) \implies x \approx y.$$

Positive MV-algebras form a peculiar quasivariety in the sense that, albeit having a logical motivation (being the quasivariety of subreducts of MV-algebras), it is not the equivalent quasivariety semantics of any logic in the sense of [2]. In this cancellative setting, we characterized the varieties of positive MV-algebras.

Theorem. The varieties of positive MV-algebras are precisely the varieties generated by a finite set of finite positive MV-algebras. Equivalently, they are precisely the varieties generated by a finite subset of $\{\mathbf{Ł}_n^+ \mid n \in \mathbb{N} \setminus \{0\}\}$, where $\mathbf{Ł}_n^+$ is the $\{\vee, \wedge, \oplus, \odot, 0, 1\}$-reduct of the $n + 1$-element MV-chain $\mathbf{Ł}_n$.

We also proved that such reducts coincide with the subdirectly irreducible finite positive MV-algebras. Using these results we show that positive MV-algebras form an unbounded sublattice of the lattice of all subvarieties of MVM.

Indeed, we prove that: a variety of positive MV-algebras is of the form $\mathcal{V}(\mathcal{K}_I)$, where I is a finite subset of $\mathbb{N}$ containing all the divisors of its elements (*divisor-closed subsets*)

Theorem. The set $\Lambda(\mathsf{MV}^+)$ of varieties of positive MV-algebras is in bijection with the set $\mathcal{J}$ of divisor-closed finite sets, as witnessed by the inverse functions:

$$f \colon \mathcal{J} \longrightarrow \Lambda(\mathsf{MV}^+) \qquad\qquad g \colon \Lambda(\mathsf{MV}^+) \longrightarrow \mathcal{J}$$
$$I \longmapsto \mathcal{V}(\mathcal{K}_I) \qquad\qquad\qquad \mathcal{V} \longmapsto \{n \in \mathbb{N} \setminus \{0\} \mid \mathbf{Ł}_n^+ \in \mathcal{V}\}.$$

where we denote by $\mathcal{K}_I$ is the set of all reducts of MV-chains with cardinality in I.

Furthermore, we present axiomatizations of all varieties of positive MV-algebras, using a strategy similar to that of Di Nola and Lettieri [3]. To do so, we define the following set of equations.

Let $I \subseteq \mathbb{N}$ be a divisor-closed set, and let m be the maximum of I (with the convention that $m = 0$ if $I = \emptyset$). We define Σ_I as the set of equations given by the single equation

$$(m + 1)x \approx mx \tag{1}$$

union the equations of the form

$$m((k - 1)x)^k \approx (kx)^m \tag{2}$$

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for all $1 \leq k \leq m$ such that $k \notin I$.

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we define the unary term $\tau_{n,k}(x)$ inductively on n as follows:

$$\tau_{0,k}(x) := \begin{cases} 1 & \text{if } k \leq -1, \\ 0 & \text{if } k \geq 0. \end{cases}$$

$$\tau_{n+1,k}(x) = \tau_{n,k-1}(x) \odot (x \oplus \tau_{n,k}(x)),$$

For every $n \in \mathbb{N}$, let Φ_n be the following set of equations, for k ranging in $\{0, \ldots, n-1\}$:

$$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x) \text{ and } \tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x). \tag{3}$$

Theorem. Let I be a divisor-closed finite set; then $\mathcal{V}(\mathcal{K}_I)$ is axiomatized by $\Phi_{\mathrm{lcm}(I)} \cup \Sigma_I$ relatively to the variety of MV-monoids, where $\mathcal{K}_I$ is the set of all reducts of MV-chains with cardinality in I.

To conclude, in the following table we summarize our axiomatizations of the almost minimal varieties of MV-monoids and the varieties of positive MV-algebras.

Variety	Axiomatization (within MV-monoids)
$\mathcal{V}(\mathbf{C}_2^\Delta)$	$x \oplus x \approx x$
$\mathcal{V}(\mathbf{C}_2^\nabla)$	$x \odot x \approx x$
$\mathcal{V}(\mathbf{L}_1^+)$	$x \oplus x \approx x$ and $x \odot x \approx x$
$\mathcal{V}(\mathbf{L}_n^+)$	$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x) \qquad$ (for $0 \leq k \leq n-1$)
	$\tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x) \qquad$ (for $0 \leq k \leq n-1$)
$\mathcal{V}(\{\mathbf{L}_n^+ \mid n \in I\})$	(setting $l := \mathrm{lcm}\, I$ and $m := \max I$)
(I div.-closed fin. set)	$\tau_{l,k}(x) \oplus \tau_{l,k}(x) \approx \tau_{l,k}(x) \qquad$ (for $0 \leq k \leq l-1$)
	$\tau_{l,k}(x) \odot \tau_{l,k}(x) \approx \tau_{l,k}(x) \qquad$ (for $0 \leq k \leq l-1$)
	$(m+1)x \approx mx$
	$m((k-1)x)^k \approx (kx)^m \quad$ (for $1 \leq k \leq m$ s.t. $k \notin I$)

References

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