Two-Layered Modal Logics: A New Beginning

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This talk presents a new approach logic with two-layered modal syntax. The syntax of these logics is given by:

- *inner* formulas build from inner variables using given *inner* propositional connectives
- *atomic outer* formulas built from inner formulas using given *modalities*
- *complex outer* formulas are built from the atomic ones using given *outer* propositional connectives

Early examples of such logics were logics of uncertainty based on Hamblin's original idea of reading the atomic outer formulas $P\varphi$ as 'probably φ' [16] and semantically interpreting it (in a given Kripke frame equipped with a probability measure) as *true* iff the probability of the set of worlds where φ is true is bigger than a given threshold. This idea was later elaborated and extended by Fagin, Halpern and many others; see e.g. [5, 15].

These initial examples used classical logic to govern the behavior of formulas on both layers. A departure from this paradigm was proposed by Hájek and Harmancová in [13] which they later developed in collaboration with Godo and Esteva in [12]. They kept classical logic to govern the inner layer of events, but proposed Łukasiewicz logic to govern the outer layer of statements on probabilities of these events. The truth degree of the atomic outer formula $P\varphi$ could then be directly identified with the probability of the set of worlds where φ is true. Later, other authors changed even the logic governing the inner layer (e.g., another fuzzy logic in order to allow for the treatment of uncertainty of vague events) or considered additional (possibly non-unary) modalities (e.g. for conditional probability), see e.g. [6–11, 14, 17].

This research thus gave rise to an interesting way of combining logics which allows to use one logic to reason about formulas (or rules) of another one with numerous examples described and developed in the literature. In our previous work [3], we took the first steps towards development of a general theory of such logics and proved, in a rather general setting, two forms of completeness theorem most commonly appearing in the literature. Although the level of generality seemed quite sufficient back then (*finitary weakly implicative logics with unit and lattice conjunction*, see [4]), recent developments in the field show the need for more: e.g., the inner logic in [2] and the outer logic in [1] are not weakly implicative, and in the former case they are not even equivalential.

In an LATD 2022 talk we presented an expansion our theory of [3] to cover even those examples, the resulting formalism was however rather cumbersome. In this talk, we propose a radical departure from the usual paradigm by taking, as elementary, the **consequence relation between equations rather than formulas.** In many situations it is just a notational variant, but in all cases it dramatically simplifies and clarifies the used formalism and the proofs on the main results. Our second contribution is a new proof of the completeness result for finitary logics which has usually involved a rather complex and cumbersome syntactical translation. In our approach, we first prove the completeness result of the related infinitary logics (for which we need no translations) and then easily transform it into the desired result for the finitary case.

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