Strong completeness for the predicate logic of the continuous t-norms

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January 30, 2025

Abstract

The axiomatic system introduced by Hájek axiomatizes first-order logic based on BL-chains. In this study, we extend this system with the axiom $(\forall x\phi)^2 \leftrightarrow \forall x\phi^2$ and the infinitary rule

$$\frac{\phi \vee (\alpha \to \beta^n) : n \in \mathbb{N}}{\phi \vee (\alpha \to \alpha\&\beta)}$$

to achieve strong completeness with respect to continuous t-norms.

In Mostowski (1961), the author proposed the study of first-order many-valued logics interpreting universal and existential quantifiers as infimum and supremum, respectively, on a set of truth values. From the 1963 article by Hay (1963), it follows that the infinitary rule

$$\frac{\phi \oplus \phi^n : n \in \mathbb{N}}{\phi}$$

can be added to the first-order Łukasiewicz calculus to obtain weak completeness with respect to the Łukasiewicz t-norm. Horn (1969) later axiomatized first-order Gödel logic in 1969.

Hájek (1998) provided a general approach to first-order fuzzy logic, introducing a syntactic logic, denoted by BL∀, which is strongly complete with respect to models based on BL-chains. However, the problem of finding an appropriate syntactic logic for models based on continuous t-norms remained unresolved.

In the propositional case, Hájek (1998) exhibited a syntactic logic that is strongly complete with respect to valuations on BL-chains. Kułacka (2018) later proved that by adding the infinitary rule

$$\frac{\phi \vee (\alpha \to \beta^n) : n \in \mathbb{N}}{\phi \vee (\alpha \to \alpha\&\beta)}$$

to the syntactic logic, a strong completeness result can be achieved with respect to valuations on t-norms.

In the first-order case, quantifiers can exhibit distinct behaviors in a continuous t-norm compared to a generic BL-chain. For example, the sentence,

$$\forall x(\phi \& \phi) \to ((\forall x \phi) \& (\forall x \phi)) \tag{RC}$$

is true in models based on continuous t-norms and is not true in general. Moreover, Hajek and Montagna (2008) demonstrated that standard first-order tautologies coincide with first-order tautologies over complete BL-chains satisfying (RC).

In this paper we show that by adding Kułacka's Infinitary Rule and Hájek and Montagna's axiom RC to BL∀, a strong standard completeness result can be proven for models based on continuous t-norms. Thus, this paper aims to contribute to the study of first-order extensions of propositional logics, such as Badia et al. (2023), Cintula et al. (2015), and Hajek and Montagna (2008).

We introduce our logic extending the logic BL∀ with an additional axiom and an infinitary rule and utilize a Henkin construction to demonstrate that for a given theory Γ and a sentence ϕ such that $\Gamma \nvdash \phi$, there exists an expanded theory Γ^* that also satisfies $\Gamma^* \nvdash \phi$ and possesses additional properties (Henkin property and *prelinearity*) necessary for the subsequent construction of a desirable Lindenbaum algebra. Then, a Lindenbaum algebra is constructed and embedded in a continuous t-norm, with the help of a new version of a weak saturation result, providing the final prerequisite for the strong completeness theorem.

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