Preservation Theorems for Many-valued Logics via Categorical Methods

James Carr[1]

University of Queensland

A canonical result in model theory is the Homomorphism Preservation Theorem (h.p.t.) which states that a first-order formula is preserved under homomorphisms on all structures if and only if it is equivalent to an existential-positive formula. A first order sentence ϕ in a vocabulary σ is *preserved under homomorphisms* iff, whenever $M \models \phi$ and there is a homomorphism of σ-structures $M \to N$ then $N \models \phi$. A sentence ψ is *existential-positive* when it is constructed using only the connectives $\wedge, \vee$ and $\exists$. The h.p.t. is an example of a preservation theorem, a family of results linking a syntactic class of formulas with preservation under a particular kind of map and which are standardly proved via compactness arguments. Rossman [6] established that the h.p.t. remains valid when restricted to finite structures.

Finite Homomorphism Preservation Theorem A first-order sentence of quantifier-rank n is preserved under homomorphisms on finite structures iff it is equivalent in the finite to an existential-positive sentence of quantifier rank $\rho(n)$ (for some explicit function $\rho : \omega \to \omega$).

This is a significant result in the field of finite model theory. It stands in contrast to other results proved via compactness, including the other preservation theorems, where the failure of the compactness also results in the failure of preservation theorem. [5] Indeed, Rossman's proof is a compactness free proof of the finite h.p.t. that can be retroactively carried out in the general case. More than providing a simple alternative proof, by avoiding compactness one maintains control of the syntactic shape of the equivalence existential positive sentence ψ. In particular this allows a comparison of the *quantifier rank* of the original sentence and its existential-positive equivalent (the ρ referenced in the theorem). In the general case this yields the *equirank* h.p.t. where the quantifier rank of the equivalent sentence is the same as original.

Adjacently, Dellunde and Vidal [4] established that a version of the h.p.t. holds for a collection of many-valued models - those defined over a fixed finite MTL-chain. In prior work [3] we showed how one can extend Rossman's proof of a finite h.p.t. to many-valued models defined over the slightly more general UL-chains, in particular establishing a finite variant to Dellunde and Vidal's result. In the many-valued setting both the notion of homomorphism and existential-positive formulas split into a number of interrelated concepts and this naturally provides a number of possible generalisations of the classical h.p.t. Many can be immediately ruled unviable by observing that the 'easy' direction of the theorem fails, i.e. that formulas in the syntactic class are not preserved by the given morphisms. The viable variant directly recoverable from the classical case links homomorphisms with existential-positive sentences ($\exists$.p) understood as they are classically. Namely homormohpsims are maps which preserve the modelling of atomic formula and existential-positive sentences those constructed from just $\wedge, \vee$ and $\exists$ (ignoring any additional algebraic connectives present in the algebraic signature).

Fixed Finite Homomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language, A UL-chain and ϕ a consistent $\mathcal{P}$ sentence over A in the finite. Then ϕ is equivalent over A in the finite to an $\exists$.p sentence ψ iff ϕ is preserved under homomorphisms. That is, there is an $\exists$.p-sentence $\psi : Mod^A_{fin}(\phi) = Mod^A_{fin}(\psi)$ iff $Mod^A_{fin}(\phi)$ is closed under homomorphisms.

This generalisation of the classical h.p.t. is one way to precisely express the idea that the 'classical part' of many-valued models still behaves classical. Indeed, the strategy to extend Rossman's result is via a 'classical counterpart' to any given many-valued model, where one demonstrates that this structure behaves well with respect to both homomorphisms and $\exists$.p-formulas. This motivates an attempt to study potential preservation theorems for many-valued models that directly address their many-valued nature. One viable preservation theorem with a more substantive many-valued character links *monomorphisms* with *strong existential-positive sentences*.

Definition Let σ be a (relational) signature and A a complete lattice. A map $f\colon M \to N$ between two σ-models is a monomorphism iff for all $R \in \sigma$ and $\bar{m} \in M$ $R^M(\bar{m}) \leq R^N(f(\bar{n}))$. A sentence ψ is said to be strong existential-positive iff it is constructed using only the existential quantifier $\exists$ and algebraic connectives $\circ$ which are order preserving in all arguments with respect to the lattice order.

A promising strategy to pursue such a result comes from recent work by Abramsky and Reggio [2]. Building on previous work on game comonads [1], where various model comparison games are studied through comands on the category of relational structures, they developed an category theoretic framework which is used to prove an abstract homomorphism preservation theorem. The general categories of interest are axiomatized as *arboreal categories* upon which abstracted notions such as game and back-and-forth system are defined. This requires a form of *resource indexing* yielding a family of subcategories for each $k \in \omega$. These transfer to an extensional category via similarly indexed family of adjunctions called a *resource indexed arboreal adjunction* (RIAA). The resulting homomorphism preservation theorem links preservation by the morphisms of the extensional category to the existence of a morphism between the adjoint images in the arboreal category (denoted $M \to_k^E N$). The classical homomorphism preservation theorem, and its finite variant due to Rossman, then emerge as one of the prime examples with the category of relational structures equipped with the Ehrenfeucht-Fraïssé RIAA. To recover the familiar h.p.t. we require an alignment of the morphism existence relation which in turn is tied to the existence of a single equivalent formula. More explicitly, for a set of models $\mathcal{D}$ we say that $\mathcal{D}$ is upwards closed with respect to a relation ∇ iff $\forall M, N \in \mathcal{D}$ if $M \in \mathcal{D}$ and $M \nabla N$ then $N \in \mathcal{D}$. The two critical lemmas are then:

Lemma 1 For all σ-structures M, N and all $k > 0$ we have $E_k(M) \to E_k(N)$ iff $M \Rightarrow^{\exists^+ FO_k} N$.

Lemma 2 For all $k \geq 0$ and all full subcategory $\mathcal{D}$ of Struct(σ) $\mathcal{D} = Mod(\psi)$ for some $\exists^+ FO_k$ iff $\mathcal{D}$ is upwards closed with respect to the relation $\Rightarrow^{\exists^+ FO_k}$, defined as for all $\psi \in \exists^+ FO_k$ $M \models \psi$ implies $N \models \psi$.

They further establish a sufficiency condition for the HP to for a given RIAA, namely the satisfaction of a series of axioms, two concerning the extensional category (E1-E2) and four concerning the RIAA adjunction itself (A1-A4). This is used in particular to establish the abstract HP for the Ehrenfeucht-Fraïssé RIAA.

In this talk we outline the attempt to adapt and apply this framework to many-valued models. This begins with an outlining of the basic behaviour of many-valued models under monomorphisms. In the classical case this category is defined relative to a propositional signature, in the many-valued case it is also defined relative to a fixed complete lattice A.

Proposition Let A be a complete lattice. We use $\mathcal{M}(\mathcal{P})$ to denote the category whose objects are $\mathcal{P}$-models defined over A. The category $\mathcal{M}(\mathcal{P})$ is complete and co-complete.

The next step is the construction of a suitably adjusted arboreal category and RIAA linking it to the category $\mathcal{M}(\mathcal{P})$.

These constructions establish that one can fit the category of many-valued models into the framework of Abramsky and Reggio. The obvious route to then establish a concrete preservation theorem for monomorphisms is to establish the statements E1-E2 and A1-A4 hold, and the two lemmas transferring the abstract HP result to the familiar one. At the time of writing these remain conjectural. As is often the case when working with many-valued models the answer is sensitive to the behaviour of the underlying algebra. Initial work suggests that when A is finite adaptions of the classical Ehrenfeucht-Fraïssé RIAA example will suffice, but when A is infinite the situation is significantly more complex.

References

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