Free algebras and coproducts in varieties of Gödel algebras

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Free Heyting algebras play a fundamental role in the study of the intuitionistic propositional calculus IPC because they arise as Lindenbaum-Tarski algebras, whose elements are equivalence classes of propositional formulas over a fixed set of variables modulo logical equivalence in IPC. Esakia duality (see [10]) proved to be a powerful tool for understanding the structure of free Heyting algebras, which are notoriously difficult to describe. Recall that a Stone space is a topological space that is compact, Hausdorff, and has a basis consisting of clopen (i.e., closed and open) subsets.

Definition 1. An *Esakia space* is a Stone space X equipped with a partial order $\leq$ such that

(i) ${\uparrow}x \coloneqq \{y \in X : x \leq y\}$ is closed for every $x \in X$,

(ii) ${\downarrow}V \coloneqq \{x \in X : x \leq y \text{ for some } y \in V\}$ is clopen for every $V \subseteq X$ clopen.

Every Esakia space X gives rise to the Heyting algebra $\mathsf{ClopUp}(X)$ of the clopen upsets of X ordered by inclusion, where $U \subseteq X$ is an *upset* if ${\uparrow}x \subseteq U$ for every $x \in U$. Vice versa, the prime spectrum $\mathsf{Spec}(H)$ of a Heyting algebra H, which is the set of the prime filters of H, becomes an Esakia space once ordered by inclusion and suitably topologized. This correspondence extends to Heyting homomorphisms between Heyting algebras and continuous p-morphisms between Esakia spaces, where $f\colon X \to Y$ is a p-morphism if $f[{\uparrow}x] = {\uparrow}f(x)$ for every $x \in X$.

Theorem 2 (Esakia duality)**.** *The category of Heyting algebras and Heyting homomorphisms is dually equivalent to the category of Esakia spaces and continuous p-morphisms.*

Different methods to study the Esakia duals of free Heyting algebras have been developed. Universal models, first investigated in [4, 19], describe the points of finite depth of the Esakia duals of finitely generated free Heyting algebras (see, e.g., [5, Sec. 3]). A different approach, known as the step-by-step method and developed in [12, 20], builds the Esakia duals of finitely generated free Heyting algebras as the inverse limits of systems of finite posets. This approach has been recently generalized beyond the finitely generated setting [2]. However, due to the complexity of free Heyting algebras, obtaining a tangible and complete description of their Esakia duals seems difficult—if not impossible—particularly for those that are free over infinitely many generators. This naturally leads us to consider free algebras in subvarieties of the variety of Heyting algebras. We will turn our attention to free Gödel algebras.

Definition 3. A *Gödel algebra* is a Heyting algebra satisfying the prelinearity axiom $(x \to y) \vee (y \to x) = 1$.

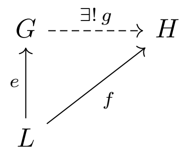
The variety GA of Gödel algebras is generated by the totally ordered Heyting algebras and provides the algebraic semantics for the superintuitionistic propositional logic known as the Gödel-Dummett logic [8]. This logic has attracted much attention, partly because it can also be regarded as a fuzzy logic (see, e.g., [3] and [18, Sec. 4.2]).

It is well known that Esakia duality restricts to a duality for Gödel algebras. An Esakia space X is called an *Esakia root system* if the order $\leq$ on ${\uparrow}x$ is total for every $x \in X$.

Proposition 4. *The category of Gödel algebras and Heyting homomorphisms is dually equivalent to the category of Esakia root systems and continuous p-morphisms.*

As all finitely generated Gödel algebras are finite [16], GA is a locally finite variety. The Esakia duals of finitely generated free Gödel algebras were described in [13], while the Esakia duals of Gödel algebras free over finite distributive lattices[1] were described in [1].

Definition 5. A Gödel algebra G is said to be free over a distributive lattice L via a lattice homomorphism $e\colon L \to G$ when the following holds: for every Gödel algebra H and lattice homomorphism $f\colon L \to H$, there is a unique Heyting homomorphism $g\colon G \to H$ such that $g \circ e = f$.



Our main result generalizes the descriptions of [1] beyond the finitely generated setting by providing a dual description of Gödel algebras free over distributive lattices, without any restriction on the cardinality of the lattice. As a consequence, we obtain a dual description of free Gödel algebras over any set of generators that generalizes the one of [13]. To provide such a description, we first recall Priestley duality for distributive lattices (see, e.g., [11]).

Definition 6. A *Priestley space* is a Stone space X equipped with a partial order $\leq$ satisfying the *Priestley separation axiom*: if $x, y \in X$ with $x \nleq y$, then there is a clopen upset U such that $x \in U$ and $y \notin U$.

The functors ClopUp and Spec generalize to a correspondence between Priestley spaces and distributive lattices, yielding Priestley duality

Theorem 7 (Priestley duality)**.** *The category of distributive lattices and lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order-preserving maps.*

We are now ready to describe the construction dual to taking the free Gödel algebra over a distributive lattice. Let X be a Priestley space. A *chain* (i.e., a totally ordered subset) of X is said to be *closed* when it is closed in the topology on X. We denote by $\mathsf{CC}(X)$ the set of all nonempty closed chains of X. Equip $\mathsf{CC}(X)$ with the Vietoris topology, which is generated by the subbasis $\{\Box V, \Diamond V \mid V \text{ clopen of } X\}$, where

$$\Box V \coloneqq \{C \in \mathsf{CC}(X) \mid F \subseteq V\} \qquad \text{and} \qquad \Diamond V \coloneqq \{C \in \mathsf{CC}(X) \mid F \cap V \neq \varnothing\}.$$

Define a partial order $\trianglelefteq$ on $\mathsf{CC}(X)$ by setting $C_1 \trianglelefteq C_2$ iff C_2 is an upset inside C_1.

The following is our main result, characterizing the Esakia duals of free Gödel algebras over distributive lattices.

Theorem 8.

1. *If X is a Priestley space, then $\mathsf{CC}(X)$ is an Esakia root system.*
2. *Let L be a distributive lattice and X its Priestley dual. Then the Gödel algebra dual to $\mathsf{CC}(X)$ is free over L.*

[1] All lattices will be assumed to be bounded and lattice homomorphisms to preserve the bounds.

Let 2 be the Priestley space consisting of the 2-element chain with the discrete topology. It is well known that for a set S, the ordered topological space 2^S with the product topology and componentwise order is a Priestley space, and that its dual distributive lattice is free over the set S. As a consequence of this observation and Theorem 8, we obtain a dual description of free Gödel algebras.

Corollary 9. *Let S be a set. Then the Gödel algebra dual to $\mathsf{CC}(2^S)$ is free over the set S.*

While products of Priestley spaces are simply cartesian products, the products in the category of Esakia spaces are difficult to describe. Consequently, computing coproducts of Heyting algebras is a nontrivial task. A generalization of the construction of universal models was employed in [14] to study the finite depth part of the product of two finite Esakia spaces, and the step-by-step method has been employed in [2] to obtain a dual description of binary products of Esakia spaces. We adapt our machinery to describe arbitrary products in the category of Esakia root systems, generalizing the description of binary products of finite Esakia root systems from [7]. As a consequence of Esakia duality, we obtain a dual description of coproducts of any family of Gödel algebras without restrictions on the cardinalities of the family and of its members.

Definition 10. Let $\{Y_i \mid i \in I\}$ be a family of Esakia root systems. Let $\prod_i Y_i$ denote the cartesian product with the componentwise order and product topology, and $\pi_i \colon \prod_i Y_i \to Y_i$ the projection onto the i-th component. We define

$$\bigotimes_{i\in I} Y_i \coloneqq \{C \in \mathsf{CC}\big(\textstyle\prod_i Y_i\big) \mid \pi_i[C] \text{ is an upset of } Y_i \text{ for every } i \in I\}$$

and equip it with the subspace topology and order induced by $\mathsf{CC}\big(\prod_i Y_i\big)$.

Theorem 11.

1. *Let $\{Y_i \mid i \in I\}$ be a family of Esakia root systems. Then $\bigotimes_{i\in I} Y_i$ is their product in the category of Esakia root systems.*

2. *Let $\{G_i \mid i \in I\}$ be a family of Gödel algebras and Y_i their Esakia duals. Then $\bigotimes_i Y_i$ is dual to the coproduct of $\{G_i \mid i \in I\}$ in GA.*

The proper subvarieties of GA form a countable chain of order type ω, and each of them is axiomatized over GA by a bounded depth axiom [9, 15] (see [17] for the corresponding characterization of the extensions of the Gödel-Dummett logic). We denote by GA_n the subvariety of GA consisting of all the Gödel algebras validating the bounded depth n axiom, and we refer to its members as GA_n-algebras. Replacing $\mathsf{CC}(X)$ with its subspace $\mathsf{CC}_n(X)$, consisting of the nonempty chains in X of size at most n, yields analogues of Theorems 8 and 11 that provide dual descriptions of free GA_n-algebras over distributive lattices and of coproducts in GA_n.

A Heyting algebra is called a *bi-Heyting algebra* if its order dual is also a Heyting algebra. The step-by-step method allows to show that every Heyting algebra free over a finite distributive lattice is a bi-Heyting algebra [12]. Using Theorem 8, we provide a characterization of the Gödel algebras free over distributive lattices that are bi-Heyting algebra. As a consequence, we deduce that free Gödel algebras are always bi-Heyting algebras. Surprisingly, we also show that the situation is very different for free GA_n-algebras.

Theorem 12.

1. *Let G be a Gödel algebra free over a distributive lattice L. Then G is a bi-Heyting algebra iff the order dual of L is a Heyting algebra.*

2. *All free Gödel algebras are bi-Heyting algebras.*

3. *A free GA_n-algebra is a bi-Heyting algebra iff it is finitely generated, and hence finite.*

These results have been collected in the manuscript [6].

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