Language for Crash Failures in Impure Simplicial Complexes

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Simplicial complexes are a well-known semantic framework in combinatorial topology to model synchronous and asynchronous distributed systems. A common type of faults considered in synchronous computation is *crash failures*. In a system with crash failures, each live process may be uncertain regarding which of the other processes have already crashed. In simplicial complexes, this is modeled semantically by considering so-called *impure* simplicial complexes. In this extended abstract, we discuss which object language is appropriate and expressive enough to reason about synchronous distributed systems with crash failures using the impure simplicial semantics.

Epistemic logic investigates knowledge and belief, and change of knowledge and belief, in multi-agent systems [5, 9, 16]. Knowledge change was extensively modeled in temporal epistemic logics [8, 13, 20] and in dynamic epistemic logics [1, 7, 19]. Epistemic logical semantics is often based on *Kripke models*, that consist of an abstract domain of global states, or worlds, between which binary relations of accessibility (or indistinguishability, depending on the agents' epistemic strength) are defined, one for each agent [17].

Combinatorial topology [14] has been used in distributed computing to model concurrency and asynchrony since [3, 10, 18], including higher-dimensional topological properties [15, 22]. Geometric manipulations such as subdivision have natural combinatorial counterparts. *Simplicial models* consist of an abstract set of *vertices* representing agents' local states. These agent-colored vertices are combined into sets called *simplices*, with a standard chromatic restriction that each simplex contain no more than one vertex per agent. Global states of the system correspond to those simplices that are maximal with respect to set inclusion and are called *facets*. *Pure* simplicial complexes correspond to distributed systems without crashes, hence, require that each facet contain exactly one vertex for each of the agents. Crashed agents are modeled by allowing facets to have fewer vertices than the total number of agents, with the understanding that all agents missing from a facet are *dead*, i.e., have crashed, whereas all agents present in the facet (as a single vertex) are *alive*. The collection of sets of vertices (simplices) in a given simplicial model is assumed to be downward closed with respect to set inclusion, with the exception of the empty set. Proper subsets of any facet are called *faces* and can be viewed as partial global states of the system.

In lieu of giving a lengthy formal definition [6], in Fig. 1 we provide examples of one pure ($\mathcal{C}_1$) and two impure ($\mathcal{C}_2$ and $\mathcal{C}_3$) simplicial models for a distributed system with three agents a, b, and c:

Each model $\mathcal{C}_i$ consists of two facets X_i and Y_i (global states) that agent a cannot distinguish, as evidenced by its vertex (local state) 0_a belonging to both. Model $\mathcal{C}_1$ is pure because its two facets (two gray triangles) X_1 and Y_1 consist of three vertices (one per agent) each. Thus, a is sure that all agents are alive and knows the value of b's variable as it is true (depicted as 1_b) in both X_1 and Y_1. On the other hand, a does not know the truth value of c's variable as it is false (0_c) in X_1 and true (1_c) in Y_1. Models $\mathcal{C}_2$ and $\mathcal{C}_3$ are impure because each contains at least one facet with strictly less than three agents: agent c is dead in X_2 of $\mathcal{C}_2$ and in X_3 of $\mathcal{C}_3$, and, additionally, agent b is dead in Y_3 of $\mathcal{C}_3$. Note that facets X_2, X_3, and Y_3 in the impure models $\mathcal{C}_2$ and $\mathcal{C}_3$ are

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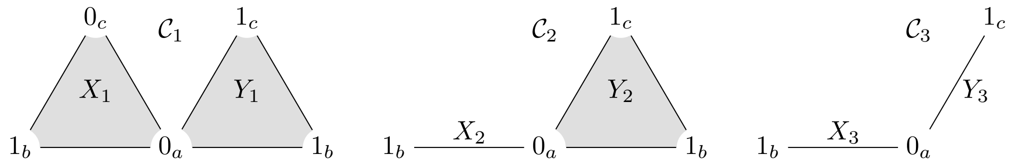

Figure 1: Impure and pure simplicial models

edges that can also be found in the pure model $\mathcal{C}_1$. However, there the corresponding edges are sides of triangles, or in simplicial complex terms, are faces of larger facets X_1 and Y_1, without being facets themselves. In both $\mathcal{C}_2$ and $\mathcal{C}_3$, agent a is unsure whether c is alive (and, additionally, whether b is alive in $\mathcal{C}_3$).

Note that we have already smuggled a small change from the standard logical language in the form of local propositional variables p_a, p_b, q_b, etc. They originate from the notion of an agent's local state in distributed systems, which is always known by the agent. Thus, a propositional variable p_a pertaining to the local state of a should be known by a, as formalized by the *locality axiom* $K_a p_a \vee K_a \neg p_a$ where modality K_a represents agent a's knowledge [6, 11]. Local variables represent a natural but not the only choice. A logic of impure simplicial complexes with standard propositional variables that are unattached to any agent (global) can be found, e.g., in [12].

We believe that a proper logic for distributed systems should include both types of variables: local variables for describing agents' local states and global variables describing global properties of the system that need not be known to any agent. For instance, asynchronous systems are typically modeled to have global time that no agent has access to, making this global time a good example of a global variable that does not belong to any agent and is, generally, not known by any agent. Logically, this would be realized by applying the locality axiom to local variables only.

The dichotomy of local and global variables is not the only choice that has been considered. Another non-trivial question regards the effect agents' crashes have on the knowledge of live agents, in particular, on their knowledge of the local variables of crashed agents. Consider again impure models $\mathcal{C}_2$ and $\mathcal{C}_3$ in Fig. 1. Does agent a know the value of, say, b's variable p_b there? The only obvious answer is that the value of p_b is known in $\mathcal{C}_2$ as it is true in both X_2 and Y_2.

But what happens with p_b in facet Y_3 of model $\mathcal{C}_3$? And what does a know about it in facet X_3? Were p_b a global variable, as in [12], its truth value would have been determined by the whole facet Y_3, and the crash of agent b would not affect it. On the other hand, there is no universally acceptable way of assigning a truth value to a local variable p_b in facet Y_3. This prompted the introduction of the third truth value 'undefined' in [6]. Propositionally, this value is treated according to the 3-valued Weak Kleene Logic, with the undefined value "infecting" any propositional formula it participates in. The question about knowledge in presence of undefined values is more subtle. In global state X_3 of model $\mathcal{C}_3$, given that p_b is undefined in Y_3, (i) should a know p_b to be true based on X_3 alone, the sole facet where p_b is defined or (ii) should a not know p_b to be true because it is not true in Y_3, which a considers possible? Both options may seem reasonable at first but option (ii) has an undesirable consequence for the dual modality $\widehat{K}_a := \neg K_a \neg$, which stands for a considers it possible. Indeed, if $\mathcal{C}_3, X_3 \nvDash K_a p_b$ according to (ii), then $\mathcal{C}_3, X_3 \vDash \widehat{K}_a \neg p_b$, i.e., agent a would have consider it possible that p_b is false despite it not being false in any facet of $\mathcal{C}_3$. This consideration explains why option (i) was chosen in [6]. It should be noted that the resulting logic is different from the way modalities work in [4].

The resulting epistemic logic of impure simplicial complexes, based on 3-valued Weak Kleene Logic on the propositional level and with local variables only, was axiomatized in [21]. The difficulty was that, as we soon discovered [2], it did not satisfy the Hennessy–Milner property for the natural notion of bisimulation. Worse than that, we have shown that no reasonable local definition of bisimulation relying on the standard back-and-forth relations would have Hennessy–Milner [21].

A failure of Hennessy–Milner often means that the language is not expressive enough. And the property lacking expressivity in terms of local variables only was quite obvious. Above, while we used the term "know", corresponding to the K_a modality for local variables, we resorted to "is sure that" regarding agents being alive or dead. The reason for this was that the latter was not expressible in the language with local variables only [2]. Hence, using "know" would have been misleading. Since one of the objectives in a distributed systems with crash failures is to reason in presence of crash failures, a language not expressive enough to talk about these crash failures in the object language is suboptimal.

Thus, based both on the desired applications and on the logical evidence of insufficient expressivity, we believe that the object language for the logic of impure simplicial complexes should include both local and global variables and that these global variables should, at the minimum, include atoms expressing that a particular agent is alive. In [2], we have shown that the logic with such atoms a for each agent a does indeed possess the Hennessy–Milner property. We are currently preparing for submission a manuscript with a complete axiom system for this logic, which extends that from [21] for local variables only.

References

[1] A. Baltag, L. S. Moss, and S. Solecki. The logic of public announcements, common knowledge, and private suspicions. In I. Gilboa, editor, *Theoretical Aspects of Rationality and Knowledge: Proceedings of the Seventh Conference (TARK 1998)*, pages 43–56. Morgan Kaufmann, 1998. URL: http://tark.org/proceedings/tark_jul22_98/p43-baltag.pdf.

[2] M. Bílková, H. van Ditmarsch, R. Kuznets, and R. Randrianomentsoa. Bisimulation for impure simplicial complexes. In A. Ciabattoni, D. Gabelaia, and I. Sedlár, editors, *Advances in Modal Logic*, volume 15, pages 225–248. College Publications, 2024.

[3] O. Biran, S. Moran, and S. Zaks. A combinatorial characterization of the distributed 1-solvable tasks. *Journal of Algorithms*, 11(3):420–440, September 1990. doi:10.1016/0196-6774(90)90020-F.

[4] S. Bonzio and N. Zamperlin. Modal weak Kleene logics: axiomatizations and relational semantics. *Journal of Logic and Computation*, Advance articles, August 2024. doi:10.1093/logcom/exae046.

[5] H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. Kooi, editors. *Handbook of Epistemic Logic*. College Publications, 2015.

[6] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*, volume 337 of *Synthese Library*. Springer, 2007. doi:10.1007/978-1-4020-5839-4.

[7] H. van Ditmarsch and R. Kuznets. Wanted dead or alive: epistemic logic for impure simplicial complexes. *Journal of Logic and Computation*, Advance articles, October 2024. doi:10.1093/logcom/exae055.

[8] C. Dixon, C. Nalon, and R. Ramanujam. Knowledge and time. In H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. Kooi, editors, *Handbook of Epistemic Logic*, pages 205–259. College Publications, 2015.

[9] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning About Knowledge*. MIT Press, 1995. doi:10.7551/mitpress/5803.001.0001.

[10] M. J. Fischer, N. A. Lynch, and M. S. Paterson. Impossibility of distributed consensus with one faulty process. *Journal of the ACM*, 32(2):374–382, April 1985. doi:10.1145/3149.214121.

[11] É. Goubault, J. Ledent, and S. Rajsbaum. A simplicial complex model for dynamic epistemic logic to study distributed task computability. *Information and Computation*, 278:104597, June 2021. doi:10.1016/j.ic.2020.104597.

[12] É. Goubault, J. Ledent, and S. Rajsbaum. A simplicial model for KB4$_n$: Epistemic logic with agents that may die. In P. Berenbrink and B. Monmege, editors, *39th International Symposium on Theoretical Aspects of Computer Science: STACS 2022, March 15–18, 2022, Marseille, France (Virtual Conference)*, volume 219 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 33:1–33:20. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.STACS.2022.33.

[13] J. Y. Halpern and Y. Moses. Knowledge and common knowledge in a distributed environment. *Journal of the ACM*, 37(3):549–587, July 1990. doi:10.1145/79147.79161.

[14] M. Herlihy, D. Kozlov, and S. Rajsbaum. *Distributed Computing through Combinatorial Topology*. Morgan Kaufmann, 2014. doi:10.1016/C2011-0-07032-1.

[15] M. Herlihy and N. Shavit. The topological structure of asynchronous computability. *Journal of the ACM*, 46(6):858–923, November 1999. doi:10.1145/331524.331529.

[16] J. Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. Cornell University Press, 1962.

[17] S. A. Kripke. A completeness theorem in modal logic. *Journal of Symbolic Logic*, 24(1):1–14, March 1959. doi:10.2307/2964568.

[18] M. C. Loui and H. H. Abu-Amara. Memory requirements for agreement among unreliable asynchronous processes. In F. P. Preparata, editor, *Parallel and Distributed Computing*, volume 4 of *Advances in Computing Research: A Research Annual*, pages 163–183. JAI Press, 1987.

[19] L. S. Moss. Dynamic epistemic logic. In H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. Kooi, editors, *Handbook of Epistemic Logic*, pages 261–312. College Publications, 2015.

[20] A. Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science*, pages 46–57. IEEE, 1977. doi:10.1109/SFCS.1977.32.

[21] R. Randrianomentsoa, H. van Ditmarsch, and R. Kuznets. Impure simplicial complexes: Complete axiomatization. *Logical Methods in Computer Science*, 19(4):3:1–3:35, October 2023. doi:10.46298/lmcs-19(4:3)2023.

[22] M. Saks and F. Zaharoglou. Wait-free k-set agreement is impossible: The topology of public knowledge. *SIAM Journal on Computing*, 29(5):1449–1483, January 2000. doi:10.1137/S0097539796307698.