A DUAL PROOF OF BLOK'S LEMMA

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1. Blok-Esakia Theorem

A *modal companion* of a superintuitionistic (si) logic **L** is any normal modal logic above **S4** in which **L** fully and faithfully embeds via the Gödel translation. The notion of a modal companion has a rich theory [6, 8, 10, 7, 4, 12, 3], culminating in a result known as the *Blok-Esakia theorem* [7, 2]. The latter states that the lattice si logics is completely isomorphic to the lattice of normal extensions of **Grz**, via the mapping that sends a si logic **L** to the least normal extension of **Grz** containing the Gödel translations of all theorems of **L**.

The literature contains several proofs of the Blok-Esakia theorem. Blok's original proof [2] is algebraic and notoriously involved (see also [11]). Esakia appears to have given a dual proof, which remains unpublished. Zacharyaschev later gave a proof using the machinery of canonical formulas [13], which Jeřábek [9] later extended to rule systems using canonical rules. More recently, Bezhanishvili and Cleani [1] (see also [5]) offered an alternative proof based on *stable* canonical rules, which are algebra-based rules that are to filtration what Jerabek's canonical rules are to selective filtration.

Our contribution is to show that the proof by Bezhanishvili and Cleani can be carried out without the machinery of stable canonical rules. Moreover, the key idea of that proof can be adapted to obtain a dual, order-topological proof that resembles Blok's original algebraic one, and gives some intuitions on it.

2. Blok's Lemma

We begin with some preliminary definitions. Let $\mathcal{H}$ be a Heyting algebra. The modal algebra $\sigma\mathcal{H}$ is constructed by expanding the free Boolean extension $B(\mathcal{H})$ of $\mathcal{H}$ with the operator

$$\Box a := \bigvee\{b \in H : b \leq a\}.$$

It is well known that $\sigma\mathcal{H}$ is always a **Grz**-algebra.

Conversely, given an **S4**-algebra $\mathcal{M}$, the *skeleton* $\rho\mathcal{M}$ of $\mathcal{M}$ is simply the Heyting algebra of open elements of $\mathcal{M}$. We recall that an element a of $\mathcal{M}$ is open when $\Box a = a$, and that the Heyting implication of $\rho\mathcal{M}$ is given by $a \to b := \Box(a \to b)$.

Dually, when $\mathcal{X}$ is an Esakia space we let $\sigma\mathcal{X} := \mathcal{X}$. Conversely, when $\mathcal{Y}$ is an **S4**-space, we let $\rho\mathcal{Y}$ be the Esakia space that results from $\mathcal{Y}$ by collapsing all clusters, and endowing the result with the quotient topology under the cluster collapse mapping. The algebraic and topological versions of the mappings σ, ρ are dual to one another.

A **Grz**-algebra $\mathcal{M}$ is called *skeletal* when it is isomorphic to $\sigma\rho\mathcal{M}$. Blok derives the Blok-Esakia theorem as a consequence of the following Lemma, now widely known as *Blok's Lemma*.

Lemma 1 (Blok's Lemma)**.** Let $\mathcal{M}$ be a **Grz**-algebra. Then $\mathcal{M} \in \mathsf{ISP_U}(\sigma\rho\mathcal{M})$.

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Let $\mathcal{M}, \mathcal{N}$ be modal algebras and let $\mathcal{A}, \mathcal{B}$ be Boolean subalgebras of $\mathcal{M}, \mathcal{N}$ respectively. A mapping $h : \mathcal{A} \to \mathcal{B}$ is called a *$\Box$-homomorphism* when it is a Boolean homomorphism and $h(\Box a) = \Box h(a)$ whenever $\Box a \in A$. The key step in Blok's proof of Lemma 1 is the following result.

Lemma 2 (Algebraic embedding lemma)**.** Let $\mathcal{M}$ be a **Grz**-algebra and let $\mathcal{N}$ be a finite Boolean subalgebra of $\mathcal{M}$. Then there is a $\Box$-embedding $h : \mathcal{N} \to \sigma\rho\mathcal{M}$.

The proof given in [1], in turn, makes use of stable maps between modal spaces:

Definition 3. Given modal spaces $\mathcal{X} = (X, R)$ and $\mathcal{Y} = (Y, R)$, a continuous function $f : \mathcal{X} \to \mathcal{Y}$ is said to be *stable* if $f(x)Rf(y)$ holds whenever xRy. Given $D \subseteq \mathsf{Clop}(\mathcal{X})$ we say that f satisfies the *bounded domain condition* (BDC) with respect to D if for all $U \in D$, if $f(x)Ry$ and $y \in U$, then there is some x' such that xRx' and $f(x') \in U$.

Given Y a finite **S4**-space, and D a domain, $\mathcal{J}(Y, D)$ denotes the *stable canonical rule*. Bezhanishvili and Cleani show that for a **Grz**-space X, $X \not\Vdash \mathcal{J}(\mathcal{Y}, D)$ implies $\rho X \not\Vdash \mathcal{J}(\mathcal{Y}, D)$, which amounts to showing the following lemma:

Lemma 4. Given a **Grz**-space $\mathcal{X} = (X, R)$, a finite **S4**-space $\mathcal{Y} = (Y, R)$, and a surjective stable map $p : \mathcal{X} \to \mathcal{Y}$ satisfying the BDC for a domain D, there is a stable surjection $p' : \rho\mathcal{X} \to \mathcal{Y}$ satisfying the BDC for the domain D.

One can then show the following:

Proposition 5. *The following statements are equivalent:*
1. *Blok's Lemma;*
2. *The algebraic stable embedding lemma;*
3. *The stable surjection lemma.*

3. Step-by-Step Proofs of Blok's Lemma

By considering the specifics of the algebraic proof of Blok's lemma and the order-topological one of [1], we will obtain a new dual proof which closely mirrors the original one of Blok. This starts with the following:

Lemma 6 (Algebraic one-step embedding)**.** Let $\mathcal{M}$ be a **Grz**-algebra and let $\mathcal{N}$ be a finite Boolean subalgebra, such that $h : \mathcal{N} \to \sigma\rho\mathcal{M}$ is a $\Box$-embedding fixing all open elements. If $x \in \mathcal{M}$ is arbitrary, then there are finitely many open elements C, and a $\Box$-embedding $h' : \langle \mathcal{N} \cup \{x\} \cup C\rangle \to \sigma\rho\mathcal{M}$ such that $h' \restriction_{\mathcal{N}} = h$.

Indeed, having this lemma, and an arbitrary finite Boolean subalgebra $\mathcal{N}$, one enumerates $\mathcal{N} = \mathcal{N}_{op} \cup \{x_1, ..., x_n\}$ where $\mathcal{N}_{op} = \langle\{\Box a : \Box a \in \mathcal{N}\}\rangle$. Then one sets $\mathcal{N}_0 = \mathcal{N}_{op}$, and successively adds one element, creating a sequence of algebras:

$$\mathcal{N}_0 \to \mathcal{N}_1 \to ... \to \mathcal{N}_n$$

where $\mathcal{N} \subseteq \mathcal{N}_n$. Having a $\Box$-embedding of the latter algebra gives us a $\Box$-embedding of $\mathcal{N}$. It is the way that such elements are picked which demands the **Grz**-axiom: given $\mathcal{N}_k$, to extend the embedding to $\langle \mathcal{N}_k \cup \{x_{k+1}\}\rangle$ one needs only define the image of x_{k+1}. This is done by picking, for each $c \in \mathcal{N}_k$, an element element w_c which is defined, by letting $u_c = \neg x_{k+1} \vee c$ as:

$$w_c := \neg(\Box(u \to \Box u) \to \Box u).$$

The use of the **Grz**-axiom lies in ensuring that $\Box(\Box(u \to \Box u) \to \Box u) \leq u$; dually, this follows if:

$$u \leq \Diamond \pi u$$

where $\pi u = u \wedge \neg\Diamond(\Diamond u \wedge \neg u)$. This is called by Esakia the *rest* of u; dually, given a clopen U, πU is called the set of *passive points of* U, and it is known that the **Grz**-axiom corresponds to every point in U being below a passive point.

Applying the key idea of the proof of [1, Main Lemma], one can obtain a dual step-by-step proof of Blok's lemma. Given a space $\mathcal{X}$, let $\rho : \mathcal{X} \to \rho\mathcal{X}$ be the cluster collapse continuous map. Moreover, a surjection $\varrho : \mathcal{X} \to \mathcal{Y}$ is called a *cluster-reducing map* when it is the quotient map induced by some equivalence relation on $\mathcal{Y}$ that never relates points belonging to different clusters in $\mathcal{X}$.

Lemma 7 (Dual one-step surjection)**.** Let $\mathcal{X} = (X, R), \mathcal{Y} = (Y, R)$ and $\mathcal{Y}' = (Y', R)$ be **S4** spaces with the following properties.

- $\mathcal{X} = (X, R)$ is a **Grz** space;
- $Y' = Y \sqcup \{\bullet\}$ and there is a cluster-reducing map $\varrho : \mathcal{Y}' \to \mathcal{Y}$ which identifies $\bullet$ with some point in its cluster, but does not identify any other points;
- There is a stable surjection $f : \mathcal{X} \to \mathcal{Y}'$ satisfying the BDC for some $D \subseteq Y'$;
- There is a stable surjection $g : \sigma\rho\mathcal{X} \to \mathcal{Y}$ satisfying the BDC for $\varrho[D]$.

Then there is a stable map $h : \sigma\rho\mathcal{X} \to \mathcal{Y}'$ satisfying the BDC for D.

We now sketch the main idea of the proof. Let $x \in Y'$ be unique with $\varrho(x) = \varrho(\bullet)$. If $y \notin \{x, \bullet\}$, we can set $h^{-1}[y] = g^{-1}[\varrho(y)]$. Note $\max(f^{-1}[\bullet])$ and $\max(f^{-1}[x])$ are disjoint and closed. Moreover, because $\mathcal{X}$ is a **Grz**-space, both these sets consist entirely of passive elements. Consequently, their images $U_\bullet := \rho[\max(f^{-1}[\bullet])]$ and $U_x := \rho[\max(f^{-1}[x])]$ are closed in $\sigma\rho\mathcal{X}$. Now, $U_\bullet$ and U_x are contained in the clopen $U := \rho(f^{-1}\{x, \bullet\})$, thus we can partition U in two clopens $V_x \supseteq U_x$ and $V_\bullet \supseteq U_\bullet$. We then put $h^{-1}(x) = V_x$ and $h^{-1}(\bullet) = V_\bullet$. So h is a stable surjection that satisfies the BDC for D, as desired.

Having obtained Lemma 7, we prove Blok's lemma following the algebraic proof strategy. Given a finite **S4**-space $\mathcal{Y} = (Y, R)$, we set $\mathcal{Y}_0 = \sigma\rho\mathcal{Y}$ and form the inverse chain

$$\mathcal{Y}_0 \leftarrow \mathcal{Y}_1 \leftarrow \dots \leftarrow \mathcal{Y}_n$$

where $\mathcal{Y}_n = \mathcal{Y}$, and $\mathcal{Y}_{i+1}$ is obtained as some cluster-expansion of $\mathcal{Y}_i$ by one additional element; the existence of the surjections in the chain being guaranteed by Lemma 7. The algebra $\mathcal{N}_0$ is the dual of $\mathcal{Y}_0$, and the posets $\mathcal{Y}_k$ – whilst not being isomorphic to the dual of $\mathcal{N}_k$ – can be seen as refinements of the decomposition given by the algebraic construction.

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