Medvedev Logic and Combinatorial Geometry

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Medvedev logic, or the *logic of finite problems*, is a well-known intermediate logic first introduced by the Russian mathematician Medvedev in his 1963 article [10]. It may be semantically defined as the logic of the Kripke frames $\{(\mathrm{Sub}\, n \smallsetminus \{\emptyset\}, \supseteq)\}_{n\in\mathbb{N}}$, i.e. the powersets $\mathrm{Sub}\, n$ of finite non-empty sets ordered by reverse inclusion, with the empty subset removed. For background and references on Medvedev logic we refer to [5]. By the *Medvedev variety* we mean the subvariety of Heyting algebras corresponding to Medvedev logic, i.e. the closure under homomorphic images, subalgebras, and products of the Heyting algebras of upper-closed subsets of the posets $\{(\mathrm{Sub}\, n \smallsetminus \{\emptyset\}, \supseteq)\}_{n\in\mathbb{N}}$.

Connections between Medvedev logic and cellular structures—notably, simplicial complexes—have long been known among specialists.[1] Indeed, the Medvedev frame $(\mathrm{Sub}\, n \smallsetminus \{\emptyset\}, \supseteq)$ is the poset of faces of an n-dimensional simplex ordered by reverse inclusion. The aim of this contribution is to initiate a systematic investigation of such connections. We discuss here two categories of central importance in combinatorial geometry, finite simplicial complexes and simplicial sets; in the manuscript [3], currently in preparation, we offer a more extensive treatment including, among others, ordered and infinite complexes, Δ-sets, and symmetric simplicial sets. It is to be hoped that these semantics based on combinatorial geometry may eventually become a further tool to tackle questions about Medvedev logic, which is notoriously difficult to analyse.

Simplicial Complexes

A classical treatment of simplicial complexes is [2]; for a contemporary account, see e.g. [6].

A (*finite*) *simplicial complex* Σ on the *set of vertices* V is a set of subsets of the finite set V such that the following conditions are satisfied.

1. Each member of Σ is non-empty.
2. For each $v \in V$, $\{v\} \in \Sigma$.
3. For each $\sigma \in \Sigma$ and for each $\emptyset \neq \tau \subseteq \sigma$, $\tau \in \Sigma$.

Let us write $\mathrm{vrt}\,\Sigma$ for the set of vertices of Σ. For simplicial complexes Σ and Δ, a *simplicial map* $\Sigma \to \Delta$ is a function $f\colon \mathrm{vrt}\,\Sigma \longrightarrow \mathrm{vrt}\,\Delta$ such that $f[\sigma] \in \Delta$ holds for each $\sigma \in \Sigma$, where $f[-]$ denotes the direct image along f. Simplicial complexes and simplicial maps form a category S.

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Subobjects of an object Σ of S are known as *subcomplexes* of Σ; they may be identified with the simplicial complexes Δ on some set of vertices $W \subseteq \operatorname{vrt}\Sigma$ such that $\delta \in \Sigma$ for each $\delta \in \Delta$. We write $\operatorname{Sub}\Sigma$ for the set of subcomplexes of the complex Σ. It is elementary that $\operatorname{Sub}\Sigma$ under inclusion order is a finite distributive lattice with top Σ and bottom the empty subcomplex. It follows $\operatorname{Sub}\Sigma$ has a unique structure of Heyting algebra. Write M for the full subcategory of Heyting algebras on those algebras isomorphic to $\operatorname{Sub}\Sigma$ for some simplicial complex Σ. The subobject functor

$$\operatorname{Sub}\colon \mathsf{S} \longrightarrow \mathsf{M}^{\mathrm{op}} \tag{1}$$

acts contravariantly on simplicial maps $f\colon \Sigma \to \Delta$ by inverse images (pullback of subobjects along f) in the standard manner. It is not difficult to prove Sub is part of a dual equivalence of categories. The explicit description of the adjoint $\mathsf{M}^{\mathrm{op}} \to \mathsf{S}$, also not difficult, is conceptually interesting in that it features the representation of simplicial complexes as a category of finite spaces and open maps; we omit details for reasons of space, and state our first result as:

Theorem 1. *The functor* (1) *is part of a dual equivalence of categories between* S *and* M*. Moreover, the variety generated by the class of objects of* M *is the Medvedev variety, and so the logic of the category* S *of finite simplicial complexes is Medvedev logic.*

Presheaf Toposes

For background on topos theory, and on presheaf toposes in particular, please see e.g. [8]. A presheaf category is a category whose objects are presheaves on a small category C, i.e. contravariant functors from C to Set, and whose maps are natural transformations between them. We denote by $\widehat{\mathsf{C}}$ the category of presheaves on C. A subpresheaf G of F in $\widehat{\mathsf{C}}$ is a subobject in the presheaf category, namely a class, up to isomorphism, of a monomorphism from G to F. Since a presheaf category turns out to be an elementary topos, it is also referred to as a presheaf topos.

A general fact concerning toposes is that, given an arbitrary topos $\mathscr{E}$ and an object X in it, the set of subobjects $\operatorname{Sub} X$ of X can be equipped with a natural structure of Heyting algebra. In case the topos is a presheaf category, for every presheaf F the Heyting algebra $\operatorname{Sub} F$ is complete, being the identity on F the top and the natural transformation from the terminal functor to F the bottom.

We will provide a criterion that is helpful in identifying the intermediate logic determined by certain presheaf toposes. By definition, the intermediate logic determined by an arbitrary topos $\mathscr{E}$ is the logic uniquely associated with the variety generated by the class of Heyting algebras $\operatorname{Sub} X$, as X ranges over all objects of $\mathscr{E}$. This coincides with the intermediate logic of all fomulæ valid in the internal Heyting algebra structure of the subobject classifier of $\mathscr{E}$, though we will not detail this fact here.

For presheaves, matters simplify. By standard general theory, the intermediate logic of a presheaf topos $\widehat{\mathsf{C}}$ is determined by subobjects of the representable functors only, i.e. by the Heyting algebras $\operatorname{Sub}(\hom(-, X))$; the elements of such algebras may in turn be identified with sieves on the object X. Building on this, Ghilardi in [7] showed how to construct out of the slice categories C/X a Kripke frame whose logic coincides with that of $\widehat{\mathsf{C}}$.

Moreover, we observe that if C admits a specific factorisation system then the Heyting algebras of subobjects of representable functors are determined by the posets of subobjects in the site C. Recall a *split epimorphism* (also called a *retraction*) in a category is an arrow $r\colon a \to b$ such that there exists a *section* $s\colon b \to a$ with $r \circ s$ the identity on b. The category C has *(split epi/mono) factorisations* if each arrow in C factors as a split epimorphism followed by a monomorphism. We prove:

Theorem 2. *Let* C *be a small category with (split epi/mono) factorisations. Then for every object* X *there is an isomorphism of Heyting algebras*

$$\mathrm{Sub}\,(\hom(-,X)) \cong \mathrm{Down}\,(\mathrm{Sub}\,X). \tag{*}$$

The right-hand side of (*) denotes the Heyting algebra of all downward-closed subsets of the poset $\mathrm{Sub}\,X$.

Hence, under the hypotheses of Theorem 2, the intermediate logic of a presheaf topos $\widehat{\mathsf{C}}$ is the logic of the opposites of the posets of subobjects $(\mathrm{Sub}\,X)^{\mathrm{op}}$, as X ranges over objects of C.

Simplicial Sets

We finally turn to the presheaf topos of simplicial sets. Any simplicial complex equipped with a partial order of its vertices that is linear on each simplex determines a simplicial set in a natural manner. In this sense simplicial sets provide a generalisation of (ordered) simplicial complexes. In fact, simplicial sets are considerably more general than simplicial complexes under several respects. Nonetheless, the intermediate logic of the presheaf topos of simplicial sets is once again Medvedev logic. For background on simplicial sets we refer e.g. to [9, 6, 8].

The category SSet of simplicial sets is the presheaf topos on the simplex category $\triangle$, namely the category of finite non-empty ordinals with morphisms the monotone functions. We denote by $[n]$ the object of $\triangle$ given by the totally ordered set with $n+1$ elements.

It is well known, and not hard to prove, that the simplex category $\triangle$ admits (split epi/mono) factorizations (for details see e.g. [9]), so Theorem 2 applies. As a consequence, we infer the intermediate logic of simplicial sets is the one determined by the posets $(\mathrm{Sub}\,[n])^{\mathrm{op}}$, as n ranges over non-negative integers. But subobjects of $[n]$ (equivalence classes of monomorphisms in $\triangle$ with codomain $[n]$) are uniquely determined by their set-theoretic images, which are the non-empty subsets of a set with $n+1$ elements.

These considerations lead to the following theorem:

Theorem 3. *The intermediate logic determined by the presheaf topos* SSet *is Medvedev logic.*

Final remark. In [1] and [4], as well as in a further forthcoming paper, the third-named author introduced and studied in collaboration with several co-authors the intermediate logic associated to classes of compact polyhedra. Without entering details, it is important to emphasise that in the framework of that research the logic determined by a simplicial complex is defined as the logic of the poset of simplices ordered by *inclusion*, as opposed to the *reverse* inclusion adopted in the present abstract. The overall picture then changes altogether. For example, it is proved in [4] that the logic of all simplicial complexes (under *inclusion* of simplices) is full intuitionistic logic, in stark contrast to Theorem 1.

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