The modal logic of classes of structures

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In [1] Hamkins and Löwe investigate how the set theoretical method of forcing between models of set theory affects the corresponding theory of a model. The method of forcing, which has become a fundamental tool in set theory, was first introduced by Paul Cohen in 1962 in order to prove the indepence of the Axiom of Choice and the Continuum Hypothesis from the other axioms of set theory. Since then, this method has widely been used to construct a huge variety of models of set theory and prove many other independence results.

With forcing one builds an extension of any model of set theory using algebraic tools; the resulting forcing extension will be another model which is closely related to the original one, but may exhibit different set theoretical truths in a way that can often be carefully controlled. Since the ground model has some access, via the forcing relation, to the truths of the forcing extension, there are clear affinities between forcing and modal logic. In fact, one can even consider the collection of all models of set theory, where the accessibility relation is induced by forcing, as an enormous Kripke model. Following this strategy, they define that a statement of set theory ϕ is possible if it holds in some forcing extension and necessary if it holds in all forcing extensions; the modal notations $\Diamond\phi$ and $\Box\phi$ express respectively that ϕ is possible and necessary.

More specifically, a **modal assertion** is a formula of propositional modal logic, which is expressed with propositional variables p_i, the usual Boolean connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\iff$ and the modal operators $\Box$, $\Diamond$. The notation $\phi(p_0, ..., p_n)$ is used to denote a modal assertion whose propositional variables are among p_0,...,p_n. A modal assertion $\phi(p_0, ..., p_n)$ is a **valid principle of forcing** if for all sentences ψ_i in the language of set theory, $\phi(\psi_0, ..., \psi_n)$ holds under the forcing interpretation of $\Box$ and $\Diamond$. We say that $\phi(p_0, ..., p_n)$ is a **ZFC provable principle of forcing** if ZFC proves all the substitution instances $\phi(\psi_0, ..., \psi_n)$. In their paper, Hamkins and Löwe prove that if ZFC is consistent, then the ZFC-provably valid principles of forcing are exactly the assertions of the well-known modal logic $\mathsf{S4.2}$: this is what the authors of [1] mean when they assert that the modal logic of forcing is $\mathsf{S4.2}$.

A natural extension of the problem introduced by Hamkins and Löwe was presented in [2]: the key idea of this paper is to consider a class of structures $\mathfrak{C}$ endowed with a binary relation $\sqsubseteq$ which is interpreted as accessibility: given $\mathbf{M}, \mathbf{N} \in \mathfrak{C}$, the notation $\mathbf{M} \sqsubseteq \mathbf{N}$ is used to state that $\mathbf{M}$ accesses $\mathbf{N}$. Clearly, this gives $(\mathfrak{C}, \sqsubseteq)$ the structure of a Kripke frame, whose Kripke models we can study. It is then natural to study the modal logic this interpretation gives rise to.

First of all, we consider the case in which $\mathfrak{C}$ is generic. Let $\mathfrak{C}$ be any class and $\sqsubseteq$ be a definable binary class relation on $\mathfrak{C}$. We consider $(\mathfrak{C}, \sqsubseteq)$ as a Kripke frame; a valuation is a function $v : Prop \times \mathfrak{C} \rightarrow \{0, 1\}$ (where we denote by $Prop$ the set of propositional variables) and a Kripke model is a triple $(\mathfrak{C}, \sqsubseteq, v)$. The **Kripke semantics** for the language $\mathcal{L}_\Box$ of modal logic can be easily defined. If $\mathbf{M} \in \mathfrak{C}$, then:

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models p$ if and only if $v(p, \mathbf{M}) = 1$ (if p is a propositional variable);

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \phi \wedge \psi$ if and only if $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \phi$ and $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \psi$;

$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \neg\phi$ if and only if $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \not\models \phi$;

$$\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \Box\phi \text{ if and only if for all } \mathbf{N} \in \mathfrak{C} \text{ such that } \mathbf{M} \sqsubseteq \mathbf{N} \text{ we have } \mathfrak{C}, \sqsubseteq, v, \mathbf{N} \models \phi.$$

A modal formula ϕ is **valid in a Kripke model** $(\mathfrak{C}, \sqsubseteq, v)$ if for every $\mathbf{M} \in \mathfrak{C}$ we have $\mathfrak{C}, \sqsubseteq, v, \mathbf{M} \models \phi$. A modal formula ϕ is **valid in a Kripke frame** $(\mathfrak{C}, \sqsubseteq)$ if it is valid in every model based on that frame. We call the **modal logic of** $(\mathfrak{C}, \sqsubseteq)$, denoted by $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$, the collection of modal formulas which are valid in $(\mathfrak{C}, \sqsubseteq)$.

The problem proposed in [2] concerns characterizing for a given frame $(\mathfrak{C}, \sqsubseteq)$ the modal logic $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ in terms of other well-known modal logics, basing on the study of the class $\mathfrak{C}$ and of the properties of the relation $\sqsubseteq$. This problem becomes more interesting when we consider $\mathfrak{C}$ as a class of structures and investigate modal logic it gives rise to. Therefore, we go on providing the general setting for classes of structures.

Let S be a non-logical vocabulary, $\mathcal{L}_S$ be the first order language with vocabulary S and let $\mathfrak{C}$ be a class of $\mathcal{L}_S$-structures (for example $\mathfrak{C}$ could be the class of all $\mathcal{L}_S$-structures satisfying a collection of $\mathcal{L}_S$-sentences). A language $\mathcal{L} \supseteq \mathcal{L}_S$ is called **$\mathfrak{C}$-adequate** if there is a definable model relation $\models$ between the elements of $\mathfrak{C}$ and $\mathcal{L}$ sentences, which extends the usual model relation od $\mathcal{L}_S$.

An **$\mathcal{L}$-translation** is a function $T : \mathit{Prop} \to \mathit{Sent}(\mathcal{L})$, assigning an $\mathcal{L}$-sentence to each propositional variable. Any $\mathcal{L}$-translation gives rise to a valuation v_T for the class $\mathfrak{C}$, called the **$\mathcal{L}$-structure valuation** in a natural way: $v_T(p, \mathbf{M}) = 1$ if and only if $\mathbf{M} \models T(p)$. Clearly, this induces a Kripke model $(\mathfrak{C}, \sqsubseteq, v_T)$.

The **$\mathcal{L}$-structure modal logic** of $(\mathfrak{C}, \sqsubseteq)$, denoted by $\mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq)$ is the set of modal formulas that are valid in each Kripke model $(\mathfrak{C}, \sqsubseteq, v_T)$ for an $\mathcal{L}$-translation T. Notice that

$$\mathsf{ML}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{ML}_{\mathcal{L}_S}(\mathfrak{C}, \sqsubseteq).$$

Now let $\mathfrak{C}$ and $\sqsubseteq$ be respectively a fixed class of structures and a binary relation on $\mathfrak{C}$. The problem of showing that $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ is some well-known modal logic L can be easily split in two tasks: proving that L is a lower bound for $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ and then showing that the lower bound is also an upper bound.

Finding a lower bound is quite easy: the strategy is based on the classical results concerning completeness of some modal logics with respect to certain classes of frames. Consider for example $\mathsf{S4.2}$, that is known to be complete with respect to the class of frames in which the accessibility relation is reflexive, transitive and directed: if we manage to prove that the relation $\sqsubseteq$ on $\mathfrak{C}$ is a directed pre-order, then we obtain $\mathsf{S4.2} \subseteq \mathsf{ML}(\mathfrak{C}, \sqsubseteq)$.

The same argument can be applied to other well-known modal logics, depending on the properties of the relation $\sqsubseteq$. In particular, if we want our logic $\mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ to be at least $\mathsf{S4}$, we need to request that $\sqsubseteq$ is reflexive and transitive on $\mathfrak{C}$. In other words, this happens if the operator which sends each $\mathbf{M} \in \mathfrak{C}$ to $\{\mathbf{N} \in \mathfrak{C} : \mathbf{M} \sqsubseteq \mathbf{N}\}$ gives rise to a closure operator.

The task of finding upper bounds and in particular proving that the lower bound is also an upper bound requires more effort and is based on finding a labeling for a fixed frame using $(\mathfrak{C}, \sqsubseteq)$. More specifically, if $(\mathcal{F}, \mathcal{R})$ is a transitive and reflexive frame with initial world w_0 (i.e. $w_0 \mathcal{R} u$ for every $u \in \mathcal{F}$), then a **$\mathfrak{C}$-labeling** of the rooted frame $(\mathcal{F}, \mathcal{R}, w_0)$ for an element $\mathbf{M} \in \mathfrak{C}$ is an assignment to each node $w \in \mathcal{F}$ of a formula ϕ_w in the language $\mathcal{L}$, such that:

1. every $\mathbf{N} \in \mathfrak{C}$ such that $\mathbf{M} \sqsubseteq \mathbf{N}$ satisfies exactly one ϕ_w;
2. if $\mathbf{N} \in \mathfrak{C}$ is such that $\mathbf{M} \sqsubseteq \mathbf{N}$ and $\mathbf{N} \models \phi_w$, then $\mathbf{N} \models \Diamond\phi_u$ if and only if $w \mathcal{R} u$;
3. $\mathbf{M} \models \phi_{w_0}$.

We can show (a proof can be found in [1]) that if for a fixed frame $(\mathcal{F}, \mathcal{R})$ satisfying the hypothesis above and for a given initial world $w_0 \in \mathcal{F}$ there exists a $\mathfrak{C}$-labeling for every $\mathbf{M} \in \mathfrak{C}$, then $\mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq)$ is contained in the modal logic of assertions valid in $\mathcal{F}$ at w_0.

Suppose now that we have managed to show that $\mathsf{L} \subseteq \mathsf{ML}(\mathfrak{C}, \sqsubseteq)$ using the strategy for lower bounds, and that $\mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{L}$ using the method for upper bounds. Then we obtain:

$$\mathsf{L} \subseteq \mathsf{ML}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq) \subseteq \mathsf{L}$$

and so $\mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq) = \mathsf{L}$.

This method provides a general strategy that can be applied to characterize $\mathsf{ML}_{\mathcal{L}}(\mathfrak{C}, \sqsubseteq)$ in terms of other well-known modal logics and it can be applied in principle to whatever class of structures we want. It could therefore be interesting to consider certain classes of algebras (for example some varities or quasivarieties) and interesting binary relations on them in order to find the modal logic they give rise to: this is exactly the framework in which the work presented in [2] lives.

The authors consider the class, which is actually a variety, of abelian groups AG together with the relation given by the operator $\mathbf{IS}$: for $\mathbf{A}$, $\mathbf{B} \in \mathsf{AG}$, the notation $\mathbf{A} \leq \mathbf{B}$ will stand for $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, i.e. $\mathbf{A}$ is isomorphic to a subgroup of $\mathbf{B}$. They manage to show that the modal logic $\mathsf{ML}(\mathsf{AG}, \leq)$ is exactly $\mathsf{S4.2}$

The fact that $\mathsf{S4.2}$ is a lower bound for the modal logic of abelian groups is clear, since it is straighforward to prove that the relation $\leq$ on AG is reflexive, transitive and directed, since given $\mathbf{A}$, $\mathbf{B} \in \mathsf{AG}$ there exists a common upper bound for them in terms of $\leq$, the cartesian product $\mathbf{A} \times \mathbf{B}$ in which both $\mathbf{A}$ and $\mathbf{B}$ can be embedded.

In order to prove that $\mathsf{S4.2}$ is also an upper bound for the modal logic of abelian groups, the strategy of finding a labeling of certain frames which are complete with respect to $\mathsf{S4.2}$ is used. Without entering into detail, it turns out that often the existence of a labeling can be broken down into simpler statements; the control statements can be seen as building blocks through which we can construct more complex statements and therefore labelings (see [3] for more details). The authors prove that given any abelian group $\mathbf{A}$, there is always the possibility of building another group $\mathbf{B}$ which satisfies exactly some specific control statements and in which $\mathbf{A}$ can be embedded, i.e. $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$. In this way they show that $\mathsf{S4.2}$ is an upper bound for $\mathsf{ML}(\mathsf{AG}, \leq)$.

The construction that appears in [2], which is based on the tools of localisations and controlled group amplifications, uses at great extent the fact that the groups are abelian: for example with non-abelian groups, there wouldn't be the possibility of dealing with controlled group amplifications, since they would not be well defined. In fact the authors leave the readers with an open question, which is related to what happens in the case of non-abelian groups.

It turns out that a different construction can be used in the general case of groups (even non-abelian ones) and, since this construction does not use the operation of inverse, it works for monoids as well. Moreover, this construction uses the same control statements as the ones that were introduced for abelian groups: this is not obvious, since control statements are defined as first order sentences in the language of the theory we are considering, which satisfy some specific axiom schemata (see [3] for more details). This means that in principle any set of formulas of the language can be chosen, provided that all the formulas in the collection satisfy some specific properties. Therefore, even if a construction does not work for a certain class and for a specific set of control statement, there may be another set of control statements and/or another construction that work for that class of structures.

In this case there is no need to look for other control statements, since the ones provided in [2] work equally well in the case of monoids, as long as another construction, which does not require commutativity and existence of inverse, is used. Therefore, using the same control statements as the ones introduced in [2] but a different construction, it can be shown that $\mathsf{ML}(\mathsf{M}, \leq) = \mathsf{S4.2}$, where we denoted by M the variety of monoids.

We highlight that this result is original and it constitutes an extension of the one presented in [2], in fact any monoid homomorphism between two groups is also a group homomorphism; suppose that $\mathbf{A}$ and $\mathbf{B}$ are groups and that $\mathbf{A}' \subseteq \mathbf{A}$ and $\mathbf{B}' \subseteq \mathbf{B}$ are monoids, then if $f : \mathbf{A}' \to \mathbf{B}'$ is a monomorphism, the natural extension of f to $\mathbf{A}$ is a group monomorphism from $\mathbf{A}$ to $\mathbf{B}$. Using our notation, if $\mathbf{A}' \leq \mathbf{B}'$ in the sense of monoids, then $\mathbf{A} \leq \mathbf{B}$ in the sense of groups. Because our purpose was to characterize the modal logic induced by the relation $\leq$ (ore equivalently by the operator $\mathbf{IS}$), this observation witnesses that the result above really extends the one about groups.

We observe that this problem could have many future developments: it could be natural to analyze what happens for other classes of structures with the relation given by $\mathbf{IS}$, or we could even switch to other significant operators, like $\mathbf{SP}$, $\mathbf{HSP}$, $\mathbf{ISP}_u$ and so on. Let's consider for example the case in which the class is a given variety V and the relation $\leq$ is induced by the operator $\mathbf{IS}$ as before: if V does not have the joint embedding property the relation is not directed and therefore it is not true that $\mathsf{S4.2}$ is a lower bound for $\mathsf{ML}(\mathsf{V}, \leq)$. However, $\leq$ is clearly reflexive and transitive whatever the variety V is, which yields $\mathsf{S4} \subseteq \mathsf{ML}(\mathsf{V}, \leq)$: this is what happens for example for lattices. Hence, one of the many possible further directions of this work could be the one of finding conditions on V which allow us to characterize the modal logic of the embedding on V in terms of modal logics which are based on $\mathsf{S4}$.

Finally we remark that the problem we deal with is different from the one that is presented in [4]: there the authors work on the collection of all the models of a fixed language and investigate the modal logic of this class with respect to the relation of being a submodel. In other words, from an algebraic point of view, they consider the class $\mathfrak{C}$ consisting of all the algebras of the same fixed type together with the relation $\geq$ such that given $\mathbf{A}, \mathbf{B} \in \mathfrak{C}$, $\mathbf{A} \geq \mathbf{B}$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$; then they try to characterize $\mathsf{ML}(\mathfrak{C}, \geq)$. This approach is different from ours mainly for two reasons: firstly, we don't work with the class of all algebras of the same type (which is very wide and contains structures that may be very different from each other), but we restrict to classes of algebras of the same type satisfying some specific axioms. Moreover the relation considered in [4] is exactly the opposite with respect to the one we deal with: for us $\mathbf{A}$ in in relation with $\mathbf{B}$ if $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, while for the authors of [4] $\mathbf{A}$ is in relation with $\mathbf{B}$ if $\mathbf{B} \in \mathbf{IS}(\mathbf{A})$.

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