Esakia Duality for Temporal Heyting Algebras

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The temporal Heyting calculus **tHC**, first presented in [3], is the natural temporal augmentation of the modalized Heyting calculus **mHC**, also first studied in [3]. It has as its algebraic models the category of temporal Heyting algebras **tHA**, a class of Heyting algebras with a "forward-looking" $\Box$ that has a left-adjoint, "backward-looking" $\blacklozenge$. Being intuitionstic, however, it lacks the modalities $\Diamond$ and $\blacksquare$ typically defined in terms of negation.

	Past	Future
$\exists$	$\blacklozenge$	$\Diamond$
$\forall$	$\blacksquare$	$\Box$

We present a suitable category of topological models for the logic (temporal Esakia spaces **tES**) and develop an Esakia duality between the categories of algebraic and topological models. This includes defining a class of filters on **tHA** and closed upsets on **tES** such that we have a poset-isomorphism between **tHA** congruences, our class of filters, and our class of closed upsets. Having achieved this, we classify simple and subdirectly-irreducible (s.i.) temporal Heyting algebras via their dual spaces as was done for "BAOs" in [4] and "distributive modal algebras" in [1]. Finally, we use this characterization to achieve a relational completeness result combining finiteness (achieved via finite model property) and a notion of "rootedness" dual to subdirect-irreducibility (analogous to the result that **IPC** is complete with respect to the class of finite trees).

Logic

Definition 1. The *modalized Heyting calculus* **mHC** is the smallest extension of **IPC** containing the following axioms and closed under modus ponens.

$$\Box(p \to q) \to (\Box p \to \Box q) \qquad p \to \Box p \qquad \Box p \to (q \vee q \to p)$$

The *temporal Heyting calculus* **tHC** is the smallest extension of **mHC** containing the following axioms and closed under modus ponens and the rule $\dfrac{\varphi \to \chi}{\blacklozenge\varphi \to \blacklozenge\chi}$.

$$\blacklozenge(p \vee q) \to (\blacklozenge p \vee \blacklozenge q) \qquad \blacklozenge\bot \to \bot \qquad p \to \Box\blacklozenge p \qquad \blacklozenge\Box p \to p$$

Algebraic models

We define the algebraic models for our logic **tHC**.

Definition 2. A *temporal Heyting algebra* is a frontal Heyting algebra $\mathbb{A}$ (see [2]) with an additional operator $\blacklozenge : \mathbb{A} \to \mathbb{A}$ such that $\blacklozenge \dashv \Box$, ie.

$$\blacklozenge a \leqslant b \quad \iff \quad a \leqslant \Box b.$$

The category **tHA** has as its objects temporal Heyting algebras and as its morphisms algebraic homomophisms.

Here we define the class of filters that will correspond to congruences on our algebras $\mathsf{Cong}(\mathbb{A})$, analogous to the correspondence between congruences and "open filters" on BAOs [5, Theorem 29]. We also define a class of elements analogous to "open elements".

Definition 3. Given $\mathbb{A} \in \mathbf{tHA}$, a $\blacklozenge$-*filter* is a filter $F \subseteq \mathbb{A}$ such that

$$a \to b \in F \quad \Longrightarrow \quad \blacklozenge a \to \blacklozenge b \in F.$$

A $\blacklozenge$-*compatible element* is an element $a \in \mathbb{A}$ such that for all b,

$$a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b).$$

The sets of $\blacklozenge$-filters and $\blacklozenge$-compatible elements of $\mathbb{A}$ are denoted by $\blacklozenge\mathsf{Filt}(\mathbb{A})$ and $\blacklozenge\mathsf{Com}(\mathbb{A})$ respectively.

Theorem 4. Given $\mathbb{A} \in \mathbf{tHA}$,

$$\langle \mathsf{Cong}(\mathbb{A}), \subseteq \rangle \cong^{\mathbf{pos}} \langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle.$$

In the finite case, the correspondence can be given element-wise.

Corollary 5. Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$,

$$\langle \mathsf{Cong}(\mathbb{A}), \subseteq \rangle \cong^{\mathbf{pos}} \langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong^{\mathbf{pos}} \langle \blacklozenge\mathsf{Com}(\mathbb{A}), \geqslant \rangle.$$

Topological models

We define the topological models for our logic **tHC**.

Definition 6. A *temporal Esakia space* is an "Rf-Heyting space" $\mathbb{X}$ (see [2]) with an additional "backward-looking" relation $R^{\triangleleft} \subseteq \mathbb{X} \times \mathbb{X}$ such that

- $R^{\triangleleft}$ is inverse to $R^{\triangleright}$ (where $R^{\triangleright}$ is the "forward-looking" relation)
- $K \in \mathsf{ClopUp}(\mathbb{X})$ implies $R^{\triangleright}[K] \in \mathsf{ClopUp}(K)$
- $R^{\triangleright}[x]$ is closed.

A *temporal Esakia morphism* is an "Rf-Heyting morphism" $f : \mathbb{X} \to \mathbb{Y}$ (see [2]) such that

$$fx_2 \; R^{\triangleleft} \; y \text{ implies } \exists x_1 \in \mathbb{X} \text{ such that } x_2 \; R^{\triangleleft} \; x_1 \text{ and } y \leqslant fx_1.$$

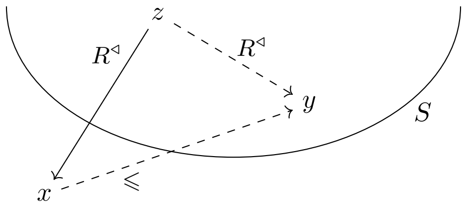
The category **tES** has as its objects temporal Esakia spaces and as its morphisms temporal Esakia morphisms.

Here we define a class of subsets on temporal Esakia spaces that will correspond to congruences of the dual algebras.

Definition 7. Given $\mathbb{X} \in \mathbf{tES}$, we call a subset $S \subseteq \mathbb{X}$ *archival* if

$$x \notin S \ni z \text{ and } z \, R^{\triangleleft} \, x \quad \Longrightarrow \quad R^{\triangleleft}[z] \cap {\uparrow}x \cap S \neq \varnothing.$$

This is depicted as follows.



We denote the set of archival subsets of $\mathbb{X}$ by $\mathsf{Arc}(\mathbb{X})$, the set of archival *upsets* of $\mathbb{X}$ by $\mathsf{ArcUp}(\mathbb{X})$, and the set of *closed* archival upsets of $\mathbb{X}$ by $\mathsf{ClArcUp}(\mathbb{X})$.

We define a notion of reachability on our spaces in terms of closed archival upsets. This is essentially analogous to the "specialization order".

Definition 8. Given $\mathbb{X} \in \mathbf{tES}$,

$$x \trianglelefteq y \quad :\Longleftrightarrow \quad y \in \bigcap \{C \in \mathsf{ClArcUp}(\mathbb{X}) \mid x \in C\}.$$

If $x \trianglelefteq y$, we say that y *is topo-reachable by* x. We denote the set of topo-roots of $\mathbb{X}$ (ie. the points that are roots with respect to $\trianglelefteq$) by $\mathsf{ToRo}(\mathbb{X})$.

In the finite case, we can define another notion of reachability (via a relation Z) only in terms of the underlying frame.

Definition 9. Given $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we define the following relation B.

$$x \, B \, w \quad :\Longleftrightarrow \quad w \leqslant x \text{ and } (w, x] \cap \mathsf{Refl}(\mathbb{X}) = \varnothing$$

We also define the following relations for all $n \in \mathbb{N}$.

$$Z_0 := \Delta_{\mathbb{X}} \qquad Z_{n+1} := Z_n ; B ; \leqslant \qquad Z := \bigcup_{m \in \mathbb{N}} Z_m$$

In the finite case, we show that these two notions of reachability are equivalent.

Proposition 10. Given $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, $x \trianglelefteq y$ iff $x \, Z \, y$. Note that this implies that x is a topo-root iff x is a Z-root as well as the fact that $\mathbb{X}$ is topo-connected iff $\mathbb{X}$ is Z-connected.

Esakia Duality

Building on the work in [2], we further augment the functors $◌_* : \mathbf{HA} \leftrightarrows \mathbf{ES} : ◌^*$ using $\blacklozenge$ and $R^{\triangleleft}$ to define eachother in the standard way. We then prove a duality between the categories of our algebraic and topological models.

Theorem 11. $\mathbf{tHA} \cong \mathbf{tES}^{\mathsf{op}}$.

We then extend our congruence-correspondence to the dual spaces.

Theorem 12. Given $\mathbb{A} \in \mathbf{tHA}$,

$$\langle \mathsf{Cong}(\mathbb{A}), \subseteq \rangle \cong^{\text{pos}} \langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong^{\text{pos}} \langle \mathsf{ClArcUp}(\mathbb{A}_*), \supseteq \rangle.$$

Characterizations

We characterize *simple* algebras lattice-theoretically and order-topologically.

Theorem 13. Given $\mathbb{A} \in \mathbf{tHA}$, the following are equivalent.

- $\mathbb{A}$ is simple
- $\blacklozenge\mathsf{Filt}(\mathbb{A}) = \{\{1\}, \mathbb{A}\}$
- $\mathbb{A}_*$ is topo-connected

In the finite case, we can give the same characterization element-wise and frame-theoretically.

Corollary 14. Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent.

- $\mathbb{A}$ is simple
- $\blacklozenge\mathsf{Com}(\mathbb{A}) = \{1, 0\}$
- $\mathbb{A}_*$ is Z-connected

We characterize *subdirectly-irreducible* algebras lattice-theoretically and order-topologically.

Theorem 15. Given $\mathbb{A} \in \mathbf{tHA}$, the following are equivalent.

- $\mathbb{A}$ is s.i.
- $\blacklozenge\mathsf{Filt}(\mathbb{A})$ has a second-least element
- $\mathsf{ToRo}(\mathbb{A}_*)$ non-empty and open

In the finite case, we can again give the same characterization element-wise and frame-theoretically.

Corollary 16. Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent.

- $\mathbb{A}$ is s.i.
- $\blacklozenge\mathsf{Com}(\mathbb{A})$ has a second-greatest element
- $\mathbb{A}_*$ is Z-rooted

Applications to tHC

We show that **tHC** has the finite model property, implying the following, stronger algebraic completeness result.

Theorem 17. The logic **tHC** is sound and complete with respect to the class of finite, subdirectly-irreducible temporal Heyting algebras $\mathbf{tHA}_{\mathsf{fsi}}$.

Finally, we use our characterization of subdirectly-irreducible temporal Heyting algebras to arrive at the following relational completeness result.

Theorem 18. The logic **tHC** is sound and complete with respect to the class of finite, Z-rooted "temporal transits" (transits [3] with an inverse relation $R^{\triangleleft}$).

References

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