C-completeness, u-presentability and Prucnal terms

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In this talk we explore a weakening of structural completeness that is obtained by considering specific subsets of the set of admissible quasiequations. We will also introduce certain terms, called *Prucnal terms* that are generalizations of the well-known concept of *ternary deductive term* introduced in [1]. We take for granted that the audience is aware of the general definition of structural completeness and primitivity and of the basic definitions and facts of universal algebra.

1 C-completeness

The concept of C-completeness has been introduced by the second author in [3]. Let A be any set; a **clone** of operations on A is a set of operations on A that contains all the projections and it is closed under composition (whenever possible). As the intersection of any family of clones is still a clone, it makes sense to talk about clone generation (that is of course a closure operator). If $\mathbf{A}$ is any algebra, then the **term clone** of $\mathbf{A}$, denoted by $\mathrm{Clo}(\mathbf{A})$, is the clone on $\mathbf{A}$ generated by all the fundamental operations.

Let now Q be a quasivariety; then the terms in the language of Q can be seen as operations on $\mathbf{F}_{\mathsf{Q}}(\omega)$, and the set of all terms is just the clone of all derived operations on $\mathbf{F}_{\mathsf{Q}}(\omega)$, i.e. the clone on $\mathbf{F}_{\mathsf{Q}}(\omega)$ generated by all the fundamental operations. We will refer to it as the **term clone** of Q and we will denote it by $\mathrm{Clo}(\mathsf{Q})$. Let C be a subclone of $\mathrm{Clo}(\mathsf{Q})$; a **C-quasiequation** is a quasiequation containing only operations from C. We say that Q is **C-structurally complete** if for every C-quasiequation Φ, if $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Phi$, then $\mathsf{Q} \vDash \Phi$. A quasivariety is **C-primitive** if all its subquasivarieties are C-structurally complete. Observe that if C' is a subclone of C and Q is C-structurally complete (C-primitive), then Q is C'-structurally complete (C'-primitive). Observe also that if T is a set of generators for C, it is easy to check that Q is C-structurally complete if and only if for every quasiequation Φ containing only operations from T, $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Phi$ entails $\mathsf{Q} \vDash \Phi$. Therefore if T is a set of terms that generates C we may talk about T-structural completeness and T-primitivity, meaning the corresponding concept for the clone generated by T. If C is the entire term clone of Q, then C-structural completeness is the usual structural completeness and C-primitivity is the usual primitivity.

Let Q be a quasivariety and let C be a subclone of the term clone of Q. Let Q^C be the class of all C-subreducts of algebras in Q; then it is easily seen that Q^C is a quasivariety in which all the C-quasiequation holding in Q are valid. So if Q^C is structurally complete or primitive, then Q is C-structurally complete or C-primitive. For instance if H is the variety of Heyting algebras, then its $\{\wedge, \rightarrow\}$-subreducts form the variety of Brouwerian semilattices, that is (as we have already observed) primitive; thus H is $\{\wedge, \rightarrow\}$-primitive.

The converse however fails to hold; the variety H of Heyting algebras is $\{\rightarrow, \neg\}$-structurally complete [5] but the quasivariety of its $\{\rightarrow, \neg\}$-subreducts is not structurally complete [2]. The problem is that there is a $\{\rightarrow, \neg\}$-quasiequation that is valid in the in $\mathbf{F}_{\mathsf{H}\{\rightarrow,\neg\}}(\omega)$ but it is not valid in $\mathbf{F}_{\mathsf{H}}(\omega)$.

2 u-presentability and Prucnal terms

Let $\mathbf{A}$ be any algebra, $\theta \in \mathrm{Con}(\mathbf{A})$ and let C be a subclone of $\mathrm{Clo}(\mathbf{A})$; by $\mathbf{A}^C$ we denote the algebra whose universe is A and whose fundamental operations are those in C. We say that θ is **u-presentable** relative to C if there is a set $\Delta \subseteq \mathrm{Con}(\mathbf{A})$ such that

1. $\theta = \bigcup \Delta$;
2. Δ is closed under finite joins;
3. $\mathbf{A}^C/\delta \in \mathbf{ISP}_u(\mathbf{A}^C)$ for all $\delta \in \Delta$;

and Δ is called a **u-presentation** of θ relative to C.

Theorem 1. *Let $\mathbf{A}$ be an algebra, $\theta \in \mathrm{Con}(\mathbf{A})$ and C a subclone of $\mathrm{Clo}(\mathbf{A})$; then the following are equivalent:*

1. *θ is u-presentable relative to C;*
2. *$\mathbf{A}^C/\theta \in \mathbf{ISP}_u(\mathbf{A}^C)$.*

Corollary 2. *For any algebra $\mathbf{A}$ and $C \subseteq \mathrm{Clo}(\mathbf{A})$ the following are equivalent:*

1. *every congruence of $\mathbf{A}$ is u-presentable relative to C;*
2. *every compact congruence of $\mathbf{A}$ is u-presentable relative to C.*

Now we can connect u-presentability and structural completeness.

Theorem 3. *Let Q be a quasivariety and C a clone of operations of Q; if every compact congruence of $\mathbf{F}_\mathsf{Q}(\omega)$ is u-presentable relative to C, then Q is C-structurally complete.*

Corollary 4. *Let Q be a quasivariety and C a clone of operations of Q; if every compact Q-congruence of every countably generated algebra in Q is u-presentable with respect to C, then Q is C-primitive.*

It is interesting to observe that if C is the entire term clone of Q, then we obtain a new necessary and sufficient condition for structural completeness.

Theorem 5. *A quasivariety Q is structurally complete if and only if every completely meet irreducible congruence $\theta \in \mathrm{Con}_\mathsf{Q}(\mathbf{F}_\mathsf{Q}(\omega))$ is u-presentable.*

Corollary 6. *A quasivariety Q is primitive if and only if for every countably generated $\mathbf{A} \in \mathsf{Q}$ every completely meet irreducible $\theta \in \mathrm{Con}_\mathsf{Q}(\mathbf{A})$ is u-presentable.*

Sometimes u-presentability is expressible directly via term operations. Let $\mathbf{A}$ be any algebra, C a subclone of the clone of all term operations on $\mathbf{A}$, T a set of generators for C and $\mathbf{A}^T$ the reduct of $\mathbf{A}$ to T; we say that $\mathbf{A}$ has the **Prucnal property** relative to C if for all $n \in \mathbb{N}$ there is a term $t_n(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ such that for any compact $\theta \in \mathrm{Con}_\mathsf{Q}(\mathbf{A})$ such that $\theta = \bigvee_{i=1}^n \vartheta_\mathbf{A}^\mathsf{Q}(a_i, b_i)$

1. the map $\sigma_n : c \longmapsto t_n(a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ is an endomorphism of $\mathbf{A}^T$;
2. $\ker(\sigma_n) = \theta$;

the terms t_n are called the **Prucnal terms** relative to C and the endomorphisms σ_n are called the **Prucnal C-endomorphisms**. If C is the entire clone of derived operations on $\mathbf{A}$, then we will drop the decoration C.

Theorem 7. *Let Q be a quasivariety and let C a clone of term operations of Q. If $\mathbf{F}_{\mathsf{Q}}(\omega)$ has the Prucnal property relative to C, then Q is C-structurally complete.*

Corollary 8. *Let Q be a quasivariety and let C a clone of term operations of Q. If every countably generated algebra in Q has the Prucnal property relative to C, then Q is C-primitive.*

3 Principal Prucnal property

A quasivariety Q has the **principal Prucnal property** relative to C, if there is a term $t(x, y, z)$ that is a Prucnal term for principal congruences, relative to C, i.e. for all $\mathbf{A} \in \mathsf{Q}$ and for all $a, b, \in A$

1. the map $\sigma : c \longmapsto t(a, b, c)$ is an endomorphism of $\mathbf{A}^C$;
2. $\ker(\sigma) = \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a, b)$.

We will show that for any quasivariety the principal Prucnal property (relative to C) implies the Prucnal property (relative to C). To this aim we need several lemmas.

Lemma 9. *[4] Let Q be a quasivariety and $\mathbf{A} \in \mathsf{Q}$; if $\theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ and $a, b \in A$ then $(\theta \vee \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a, b))/\theta = \vartheta_{\mathbf{A}/\theta}(a/\theta, b/\theta)$.*

For the following lemma, in order to avoid clutter we use a special notation; first we will denote a sequence $a_1, \dots, a_n \in A$ by $\mathbf{a}^n$. Next if $\theta \in \mathrm{Con}(\mathbf{A})$ we will write $\bar{\mathbf{A}}$ for $\mathbf{A}/\theta$, $\overline{a}$ for a/θ and $\overline{\mathbf{a}}^n$ for $\overline{a}_1, \dots, \overline{a}_n$.

Lemma 10. *Let Q be a quasivariety, $\mathbf{A} \in \mathsf{Q}$ and $a_1, \dots, a_n, b_1, \dots, b_n \in A$. If $\overline{x} = x/\vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_n, b_n)$, $\bar{\mathbf{A}} = \mathbf{A}/\vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_n, b_n)$ and $c, d \in A$ then*

$$\begin{aligned} &(c, d) \in \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_1, b_1) \vee \dots \vee \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_n, b_n) \quad \textit{if and only if} \\ &(\overline{c}, \overline{d}) \in \vartheta^{\mathsf{Q}}_{\bar{\mathbf{A}}}(\overline{a}_1, \overline{b}_1) \vee \dots \vee \vartheta^{\mathsf{Q}}_{\bar{\mathbf{A}}}(\overline{a}_{n-1}, \overline{b}_{n-1}). \end{aligned}$$

Let Q be a quasivariety with a principal Prucnal term relative to C, say $t(x, y, z)$. We define for $n \geq 1$

$$\begin{aligned} &t_1(x_1, y_1, z) := t(x_1, y_1, z) \\ &t_{n+1} = t_n(x_1, \dots, x_n, y_1, \dots, y_n, t(x_n, y_n, z)). \end{aligned}$$

Lemma 11. *Let Q be a quasivariety with principal Prucnal term $t(x, y, z)$ relative to C, $\mathbf{A} \in \mathsf{Q}$ and $a, b, c, d \in A$. Then*

$$\begin{aligned} &(c, d) \in \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_1, b_1) \vee \dots \vee \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a_n, b_n) \quad \textit{if and only if} \\ &t_n(\mathbf{a}^n, \mathbf{b}^n, c) = t_n(\mathbf{a}^n, \mathbf{b}^n, d). \end{aligned}$$

We observe that Lemma 11 is a generalization of Theorem 2.6 in [1].

Corollary 12. *If a quasivariety Q has a principal Prucnal term relative to C, then it has the Prucnal property relative to C.*

Proof. The terms $t_n, n \geq 1$ clearly satisfy the first condition for Prucnal terms, as iterated compositions of t. By Lemma 11 they also satisfy the second. □

4 Ternary deduction terms

Here we will consider a special principal Prucnal term. Let Q be a quasivariety; a **ternary deductive term (TD-term)** for Q is a ternary term $t(x, y, z)$ such that

- $\mathsf{Q} \vDash t(x, x, z)$;
- if $(c, d) \in \vartheta^{\mathsf{Q}}_{\mathbf{A}}(a, b)$ then $t(a, b, c) = t(a, b, d)$.

Let Q be a quasivariety with a TD-term $t(x, y, z)$ and q a k-term of Q; we say that q **commutes with** t if for all $\mathbf{A} \in \mathsf{Q}$ and $a, b, c_1, \ldots, c_k \in A$

$$t(a, b, q(c_1, \ldots, c_k)) = q(t(a, b, c_1), \ldots, t(a, b, c_k)).$$

Theorem 13. *A quasivariety haveing a TD-term is a variety.*

Theorem 14. *If $t(x, y, z)$ is a TD-term for Q, then t is a principal Prucnal term relative to any clone C of operations that commute with $t(x, y, z)$.*

Corollary 15. *Let Q be a variety with a TD-term. Then for all nontrivial $\mathbf{A} \in \mathsf{Q}$, $\mathbf{A}$ has the Prucnal property for $\mathbf{A}$, relative to any clone C of operations that commute with the relative TD-term.*

Corollary 16. *Let Q be a quasivariety with a relative TD-term; then Q is C-primitive for any clone C of terms that commute with the relative TD-term.*

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