The algebraic structure of spaces of integrable functions

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Abstract

We characterize the functions $\mathbb{R}^I \to \mathbb{R}$ that preserve integrability, meaning that their pointwise application maps any I-tuple of integrable functions to an integrable function (as, for example, the sum $+\colon \mathbb{R}^2 \to \mathbb{R}$). We show that Dedekind σ-complete truncated vector lattices are precisely the algebras with integrability-preserving functions as function symbols and that satisfy all equations true in $\mathbb{R}$. We also show that an analogous study restricted to finite measure spaces gives the class of Dedekind σ-complete vector lattices with weak unit. Furthermore, we provide concrete models for free algebras in these categories.

1 Introduction

1.1 Operations that preserve integrability

We investigate the operations that are somehow implicit in the theory of integration by addressing the following question: which operations preserve integrability, in the sense that they return integrable functions when applied to integrable functions?

To clarify the question, we recall some definitions.

For $(\Omega, \mathcal{F}, \mu)$ a measure space (with the range of μ in $[0, +\infty]$), we write

$$\mathcal{L}^1(\mu) \coloneqq \left\{ f\colon \Omega \to \mathbb{R} \mid f \text{ is } \mathcal{F}\text{-measurable and } \int_\Omega |f| \, d\mu < \infty \right\}.$$

It is well known that, for $f, g \in \mathcal{L}^1(\mu)$, we have $f + g \in \mathcal{L}^1(\mu)$. In other words, $\mathcal{L}^1(\mu)$ is closed under the pointwise addition, induced by the addition of real numbers $+\colon \mathbb{R}^2 \to \mathbb{R}$. More generally, consider a set I and a function $\tau\colon \mathbb{R}^I \to \mathbb{R}$, which we shall call an *operation of arity* $|I|$. We say that $\mathcal{L}^1(\mu)$ *is closed under* τ if τ returns functions in $\mathcal{L}^1(\mu)$ when applied to functions in $\mathcal{L}^1(\mu)$; i.e., for every $(f_i)_{i\in I} \subseteq \mathcal{L}^1(\mu)$, the function $\tau((f_i)_{i\in I})\colon \Omega \to \mathbb{R}$ given by $x \in \Omega \mapsto \tau((f_i(x))_{i\in I})$ belongs to $\mathcal{L}^1(\mu)$. If $\mathcal{L}^1(\mu)$ is closed under τ, we also say that τ *preserves integrability over* $(\Omega, \mathcal{F}, \mu)$. Finally, we say that τ *preserves integrability* if τ preserves integrability over every measure space.

Our first contribution is a solution to the following question, mentioned at the beginning.

Question 1. Under which operations $\mathbb{R}^I \to \mathbb{R}$ are $\mathcal{L}^1$ spaces closed? Equivalently, which operations preserve integrability?

For every $n \in \mathbb{N}$, we prove that a function $\tau\colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability precisely when τ is Borel measurable and *sublinear*, meaning that there are positive real numbers $\lambda_0, \dots, \lambda_{n-1}$ such that, for every $x \in \mathbb{R}^n$,

$$|\tau(x)| \leq \sum_{i=0}^{n-1} \lambda_i |x_i|.$$

We also prove an analogous result for the general case of arbitrary arity (not only for finite n), settling Question 1.

Examples of such operations are the constant 0, the addition $+$, the binary maximum $\vee$ and minimum $\wedge$, and, for $\lambda \in \mathbb{R}$, the scalar multiplication $\lambda(\,\cdot\,)$ by λ. A further example is the operation of countably infinite arity $\curlyvee$ defined as

$$\curlyvee(y, x_0, x_1, \dots) \coloneqq \sup_{n \in \omega}\{\min\{y, x_n\}\}.$$

Yet another example is the unary operation

$$\begin{aligned} \overline{\,\cdot\,} \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \overline{x} \coloneqq x \wedge 1, \end{aligned}$$

called *truncation*. Here, although the constant function 1 may fail to belong to $\mathcal{L}^1(\mu)$, it is always the case that $f \in \mathcal{L}^1(\mu)$ implies $\overline{f} \in \mathcal{L}^1(\mu)$.

Non-examples are the binary product and any non-zero constant.

We prove that the examples above are essentially "all" the operations that preserve integrability, in the sense that every operation that preserves integrability may be obtained from these by composition.

Moreover, we address a variation of Question 1 in which we restrict attention to finite measures.

In particular, for every $n \in \mathbb{N}$, a function $\tau \colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability over finite measure spaces precisely when it is Borel measurable and *subaffine*, meaning that there are positive real numbers $\lambda_0, \dots, \lambda_{n-1}, k$ such that, for every $x \in \mathbb{R}^n$,

$$|\tau(x)| \leq k + \sum_{i=0}^{n-1} \lambda_i |x_i|.$$

Furthermore, we prove that the operations that preserve integrability over finite measure spaces can be obtained by composition from 0, $+$, $\vee$, $\wedge$, $\lambda(\,\cdot\,)$ (for each $\lambda \in \mathbb{R}$), $\curlyvee$ and the constant 1 (which replaces the truncation operation $\overline{\,\cdot\,}$).

1.2 Truncated vector lattices and weak units

We investigate the equational laws satisfied by the operations that preserve integrability. We therefore work in the setting of varieties of algebras [BS81]. Under the term *variety* we include also infinitary varieties, i.e. varieties admitting primitive operations of infinite arity.

We assume familiarity with the basic theory of vector lattices, also known as vector lattices. All needed background can be found, for example, in the standard reference [LZ71]. As usual, for a vector lattice G, we set $G^+ \coloneqq \{x \in G \mid x \geq 0\}$.

A *truncated vector lattice* is a vector lattice G endowed with a function $\overline{\,\cdot\,} \colon G^+ \to G^+$, called *truncation*, which has the following properties for all $f, g \in G^+$.

(B1) $f \wedge \overline{g} \leq \overline{f} \leq f$.

(B2) If $\overline{f} = 0$, then $f = 0$.

(B3) If $nf = \overline{nf}$ for every $n \in \omega$, then $f = 0$.

The notion of truncation is due to R. N. Ball [Bal14], who introduced it in the context of lattice-ordered groups.

Let us say that a partially ordered set B is *Dedekind σ-complete* if every nonempty countable subset $A \subseteq B$ that admits an upper bound admits a supremum. We prove that the category of Dedekind σ-complete truncated vector lattices is a variety generated by $\mathbb{R}$. This variety can be presented as having operations of finite arity together with the single operation $\curlyvee$ of countably infinite arity. Moreover, we prove that the variety is finitely axiomatisable by equations over the theory of vector lattices. One consequence is that the free Dedekind σ-complete truncated vector lattice over a set I (exists, and) is

$$\left\{f \colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ is measurable and sublinear}\right\}.$$

We prove analogous results for operations that preserve integrability over *finite* measure spaces. An element 1 of a vector lattice G is a *weak (order) unit* if $1 \geq 0$ and, for all $f \in G$, $f \wedge 1 = 0$ implies $f = 0$. We prove that the category of Dedekind σ-complete vector lattices with weak unit is a variety generated by $\mathbb{R}$, again with primitive operations of countable arity. It, too, is finitely axiomatisable by equations over the theory of vector lattices. We show that the free Dedekind σ-complete vector lattice with weak unit over a set I (exists, and) is

$$\left\{f \colon \mathbb{R}^I \to \mathbb{R} \mid f \text{ is measurable and subaffine}\right\}.$$

The above presentations of the free algebras depend on a version of the Loomis-Sikorski Theorem for vector lattices, whose proof can be found in [BvR97] (and can also be recovered from the combination of [BdPvR08] and [BvR89]). The theorem and its variants have a long history: for a fuller bibliographic account, please see [BdPvR08].

This presentation is based on [Abb20b] (and partly on [Abb20a]).

References

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